ON THE LEGO-TEICHMÜLLER GAME FOR FINITE G-COVER

TANVIR PRINCE

ABSTRACT. Given a smooth, oriented, closed surface Σ of genus zero, possibly with boundary, let $\tilde{\Sigma} \longrightarrow \Sigma$ be a given G-cover of Σ, where G is a given finite group. Let S_n denote the standard sphere with n holes. There are many ways of gluing together several G-cover of S_n to construct the G-cover $\tilde{\Sigma} \longrightarrow \Sigma$, of Σ. We let $\mathrm{M}(\tilde{\Sigma}, \Sigma)$ be the set of all ways to construct the given G-cover, $\tilde{\Sigma} \longrightarrow \Sigma$, of Σ from gluing of several G-covers of S_n, here n may vary. In this paper, we define some simple moves and relation which will turn $\mathrm{M}(\tilde{\Sigma}, \Sigma)$ into a connected and simply-connected complex. This will be used in the future paper to construct G-equivariant modular functor. This G-equivariant modular functor will be an extension of the usual modular functor.

CONTENTS

1. Introduction

This paper can be thought of as an extension of the paper "On the Lego-Teichmüller Game" by Bojko Bakalov and Alexander Kirillov Jr [BK]. In the paper [BK], authors gave the description of how to represent a given smooth, oriented surface, possibly with boundary by gluing of several "simple pieces". Here "simple pieces" are just sphere with n holes. One application of this is the construction of modular functor; it suffices to define the vector spaces associated with sphere with n holes. Then since the behavior of modular functor under gluing is known, this defines a unique vector space associated to any surface. Of course, there are many different ways to represent a given surface as a result of gluing several spheres with n holes, here n can vary. So some natural questions arise; like how one can describe different ways of gluing "standard pieces" that give the same surface? Let $M(\Sigma)$ be the set of all such way of getting the surface Σ from the "simple pieces". In the paper [BK], they described some simple moves or edges and some relations among them which turned $M(\Sigma)$ into a connected and simply-connected complex.

This definition of modular functor can be extended. Let G be a finite group. And let $\tilde{\Sigma} \longrightarrow \Sigma$ be a given G-cover of Σ. It is possible to extend the definition of modular functor from the surface, Σ, to the G-covers of surface, $\tilde{\Sigma} \longrightarrow \Sigma$. This idea will be formalized in later paper. This extended modular functor will also satisfy a similar gluing axiom, just like the regular modular functor, but now we are gluing G-covers of surface and not just surface. Thus if we know the value of the extended modular functor on the "simple pieces", of course we need to know what are these "simple pieces" in this case, then this will be enough to define Extended modular functor to any G-cover. Since any G-cover can be constructed from the gluing of these "simple pieces" and the behavior of the extended modular functor under gluing is known, this will define a unique value for a given G-cover.

As in the paper of [BK], we are faced with similar questions. Let $M(\tilde{\Sigma}, \Sigma)$ denote the set of all possible way of gluing together "simple pieces" to construct the given G-cover, $\tilde{\Sigma} \longrightarrow \Sigma$. Is it possible to define some simple moves to go from one parameterization to the other? Is is possible to define all the relation between these moves? ie describe when a sequence of moves applied to a given parameterization yields the same parameterization? In other word, we are trying to define some simple moves and relations similar to the paper [BK], which will turn $M(\tilde{\Sigma}, \Sigma)$ into a connected and simply-connected complex. This paper runs side by

side with the paper [BK], although there are some differences. It is recommended that readers first read the paper [BK] before reading this one. This paper only deals with the case when the base surface, Σ, has genus zero. The case of positive genus will be considered in the subsequent papers.

2. Some basic definition

Let us start with some basic definitions.

2.1. **Extended Surface.**

Definition 1. An extended surface is a compact, smooth, oriented, closed surface, Σ, possibly with boundary and also comes with a choice of distinguished or marked points on each of its boundary component.

We denote by $A(\Sigma)$, the set of the boundary components. So an extended surface will be denoted by $(\Sigma, \{p_a\}_{a \in A(\Sigma)})$ where p_a is the choice of marked point on the a th boundary component.
Sometimes we will also denote a boundary circle by a Greek letter.

2.2. G-**cover of Extended surface .** G will always denote a finite group, which is given and fixed throughout the whole paper. Let $(\Sigma, \{p_a\}_{a \in A(\Sigma)})$ be an extended surface.

Definition 2. By a G cover of $(\Sigma, \{p_a\}_{a \in A(\Sigma)})$, we mean $(\pi : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ where $(\pi : \tilde{\Sigma} \longrightarrow \Sigma)$ is a principal G-cover and $\{\tilde{p_a}\}$ are choice of points on the fiber of p_a. In other word, $\tilde{p_a} \in \pi^{-1}(p_a)$ for all $a \in A(\Sigma)$.

2.3. **Morphism between G-cover of extended surface .**

Definition 3. Given two G covers of $(\Sigma, \{p_a\})$ say $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ and $(\hat{\pi} : \hat{\Sigma} \longrightarrow \Sigma, \{\hat{p_a}\})$, by a morphism between them we mean a homeomorphism $f : \tilde{\Sigma} \to \hat{\Sigma}$ so that the following conditions are satisfied:

i) $f(\tilde{p_a}) = \hat{p_a}$ for all $a \in A(\Sigma)$
ii) $\hat{\pi} f = f_* \tilde{\pi}$, here $f_* : \Sigma \to \Sigma$ is the homeomorphism we get by restriction of f to Σ
iii) f preserves the action of G on each fiber.

See the diagram below:

$$
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{f} & \hat{\Sigma} \\
{\scriptstyle \tilde{\pi}} \downarrow & & \downarrow {\scriptstyle \hat{\pi}} \\
\Sigma & \xrightarrow{f_*} & \Sigma
\end{array}
$$

Remark 1. Sometimes, we require $f_* : \Sigma \to \Sigma$ to be the identity.

Although in this definition, we leave the base space fixed, there is no need to do this. We can easily defined morphism between two G covers where the base space is not fixed.
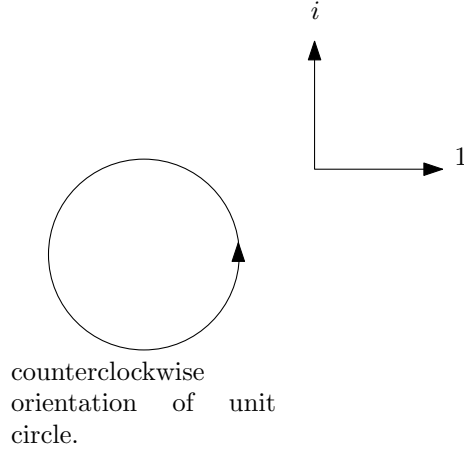
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counterclockwise orientation of unit circle.

FIGURE 1. orientation on the complement of the unit disk

Definition 4. Let $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ be a G cover of $(\Sigma, \{p_a\})$ and let $(\hat{\pi} : \hat{\Sigma} \longrightarrow \Sigma', \{\hat{p_a}\})$ be a G cover of $(\Sigma', \{p'_a\})$. By a morphism between them, we mean a homeomorphism $f : \tilde{\Sigma} \to \hat{\Sigma}$ so that it satisfies the following condition:
i) $f(\tilde{p_a}) = \hat{p_a}$ for all $\tilde{p_a}$
ii) f commutes with the action of G.

Because of the second condition, it is easily seen that f descends to a homeomorphism on the base surface by $f_* : \Sigma \to \Sigma'$ where we defined $f_*(b) = \hat{\pi} f(\tilde{b})$ where $\tilde{b} \in \tilde{\pi}^{-1}(b)$. It is easily seen that f_* does not depend on the choice of $\tilde{b}$. Also $\hat{\pi} f = f_* \tilde{\pi}$.
In other words the diagram below is commutative:

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{f} & \hat{\Sigma} \\ \tilde{\pi} \downarrow & & \downarrow \hat{\pi} \\ \Sigma & \xdashrightarrow{f_*} & \Sigma' \end{array}$$

Remark 2. Here we introduce the notation, $\mathrm{Mor}_\Sigma(\tilde{\Sigma}, \tilde{\Sigma})$ to denote all the morphisms between the G-cover $\tilde{\Sigma}$ and $\tilde{\Sigma}$ so that the induced map on Σ is identity. Also we use the notation $\mathrm{Mor}(\tilde{\Sigma}, \hat{\Sigma})$ to denote all morphism between G-cover $\tilde{\Sigma}$ and $\hat{\Sigma}$ where the induced map on the base surface can be anything.

2.4. **Orientation of the boundary circle.** The orientation of the extended surface, Σ, naturally induces orientation on the boundary circle. We want to explain this in a little detail. Let $D = \{z \in \mathbb{C} | |z| < 1\}$. First, we fix an orientation on the complement of D. This orientation is given by the choice of the basis, $\{1, i\}$, on $\mathbb{C}$, and the counterclockwise orientation on the unit circle. See the figure 1. Now for each boundary circle of Σ, we choose a small neighborhood around the circle. This neighborhood is homeomorphic to $E = \{z \in \mathbb{C} | 1 \leq |z| \leq 1 + \epsilon\}$ for some ϵ. For any such homeomorphism which preserve the orientation of the surface, there is an

unique choice of orientation of the boundary circle of Σ so that the map also preserve the orientation on the boundary. This gives an orientation to each boundary component of Σ.

2.5. **Monodromy** .

2.5.1. *Definition of Monodromy.* Monodromy will be an important tool to describe G-cover. In fact soon we will prove that two G-covers of a surface of genus 0 are isomorphic iff their monodromy is same.

Definition 5.   : [Monodromy around a boundary circle:] Let $(\pi : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ be a G-cover of $(\Sigma, \{p_a\}_{a \in A(\Sigma)})$. Consider the a th boundary circle, S, where the base point on S is p_a and the point on the fiber above is $\tilde{p_a}$. This S has an orientation which comes from the orientation of the surface. Let $\alpha : [0,1] \to S$ be a parameterization of the boundary circle S which also preserves the orientation of S. We also assume that $\alpha(0) = \alpha(1) = p_a$. Then there is a unique lifting of α to the G-cover, say $\tilde{\alpha} : [0,1] \to \tilde{\Sigma}$ such that $\tilde{\alpha}(0) = \tilde{p_a}$. We define the monodromy, $m \in G$, of this a th boundary circle to be that unique element of G such that

$$(1) \qquad\qquad\qquad m\tilde{\alpha}(0) = \tilde{\alpha}(1)$$

Lemma 1. *Monodromy does not depend on the choice of parameterization*

Proof. This is not hard to see. From the definition, we see that the monodromy only depends on the starting point of the lifting; if we know the starting point of the lifting, everything else is determined by the cover, including the end point of the lifting. And monodromy only depends on the starting point and the ending point of the lifting. The details are left for the reader. $\qquad\square$

2.5.2. *Relation Between Monodromy and Lifting of a Map.* Given two G-covers and a map between their base surfaces, we want to know under what condition this map can be lifted to the covers. Monodromy helps us to partially answer this question. We have the following lemma.

Lemma 2. *Let $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ be a G cover of $(\Sigma, \{p_a\})$ and let $(\hat{\pi} : \hat{\Sigma} \longrightarrow \Sigma', \{\hat{p_a}\})$ be a G cover of $(\Sigma', \{p'_a\})$. Here we also assume the base surfaces to be connected. Let $f : \Sigma \to \Sigma'$ be a homeomorphism of the base surfaces which maps marked points to marked points, that is $f(p_a) = p'_a$. See the diagram below:*

$$
\begin{array}{ccc}
\tilde{\Sigma} & \xdashrightarrow{\tilde{f}} & \hat{\Sigma} \\
\tilde{\pi} \downarrow & & \downarrow \hat{\pi} \\
\Sigma & \xrightarrow{f} & \Sigma'
\end{array}
$$

Then

  (1) *The lifting of f to G-covers $\tilde{\Sigma} \longrightarrow \hat{\Sigma}$ is unique, if it exists at all.*
  (2) *If in addition, we assume that the genus of both base surfaces are 0, then f can be lifted iff monodromy of the two G-covers match. That is*

$$m((\partial\Sigma)_i) = m(f(\partial\Sigma)_i)$$

where $m((\partial\Sigma)_i) = $ monodromy of $(\partial\Sigma)_i$ boundary circle of Σ.
and $m(f(\partial\Sigma)_i) = $ monodromy of $f(\partial\Sigma)_i$ boundary circle of Σ'.
Note that the homeomorphism, $f : \Sigma \to \Sigma'$, maps the boundary components
of Σ to the boundary components of Σ'.

Proof. (1) is obvious. To be more specific, let f_1 and f_2 be two lifting of f. From the definition of lifting and morphism of G covers (see 2.3), f_1 and f_2 must satisfy

$$\hat{\pi} f_1 = \hat{\pi} f_2 = f\tilde{\pi}$$

So if $x \in \Sigma$ then both f_1 and f_2 maps the fiber above the x to the fiber above the $f(x)$. Moreover from the definition of morphism between G-cover we must have $f_1(\tilde{p_a}) = f_2(\tilde{p_a}) = \hat{p_a}$
Also both f_1 and f_2 commute with the action of G. This information guarantees that they must agree on all the fibers of same connected component. But since our base surfaces are connected they must agree everywhere. So $f_1 = f_2$.

For (2), we will use the following standard proposition of G cover:

Proposition 1. *Let $\tilde{\pi} : \tilde{\Sigma} \to \Sigma$ and $\hat{\pi} : \hat{\Sigma} \to \Sigma'$ be two G covers and $f : \Sigma \to \Sigma'$ be a map between the base surface. Then this map, f can be lifted to the G cover if and only if the monodromy of every loop, α, in $\tilde{\pi} : \tilde{\Sigma} \to \Sigma$ is equal to the monodromy of the corresponding loop, $f(\alpha)$, in $\hat{\pi} : \hat{\Sigma} \to \Sigma'$. And in the case that both of the base surfaces, Σ and Σ', have genus zero, then f can be lifted to the G covers if and only if the monodromy of the corresponding boundary circles match.*

Proof. A proof of this or some equivalent statements can be found in many standard books on topology. For example see section 1.3 of [AH]. □

From the above proposition, the statement 2 of our lemma easily followed.

□

2.6. **Gluing of two G-covers of extended surfaces.**

Definition 6. Let $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ be a G cover of $(\Sigma, \{p_a\})$ and let $(\hat{\pi} : \hat{\Sigma} \longrightarrow \Sigma', \{\hat{p_a}\})$ be a G cover of $(\Sigma', \{p'_a\})$. Let $s_{i,j} : (\partial\Sigma)_i \to (\partial\Sigma')_j$ be an orientation reversing map of the ith boundary circle of Σ to the j th boundary circle of Σ' so that $s_{i,j}(p_i) = p'_j$. Then we define the gluing of these two G covers under $s_{i,j}$ to be the following G cover:

$$\tilde{\pi} \bigsqcup_{s_{i,j}} \hat{\pi} : \tilde{\Sigma} \bigsqcup_{s_{i,j}} \hat{\Sigma} \longrightarrow \Sigma \bigsqcup_{s_{i,j}} \Sigma'$$

where $\Sigma \bigsqcup_{s_{i,j}} \Sigma'$ is the surface obtained by identifying points on $(\partial\Sigma)_i$ to the points on $(\partial\Sigma')_j$ through the map $s_{i,j}$. And $\tilde{\Sigma} \bigsqcup_{s_{i,j}} \hat{\Sigma}$ is the G cover obtained by identifing a point (t, g), where $t \in (\partial\Sigma)_i$ and $g \in G$, to the point $(s_{i,j}(t), g)$.

2.6.1. *How to glue two G-covers.* Such a gluing not always exists, but if it exists, it is unique. So we have the following lemma.
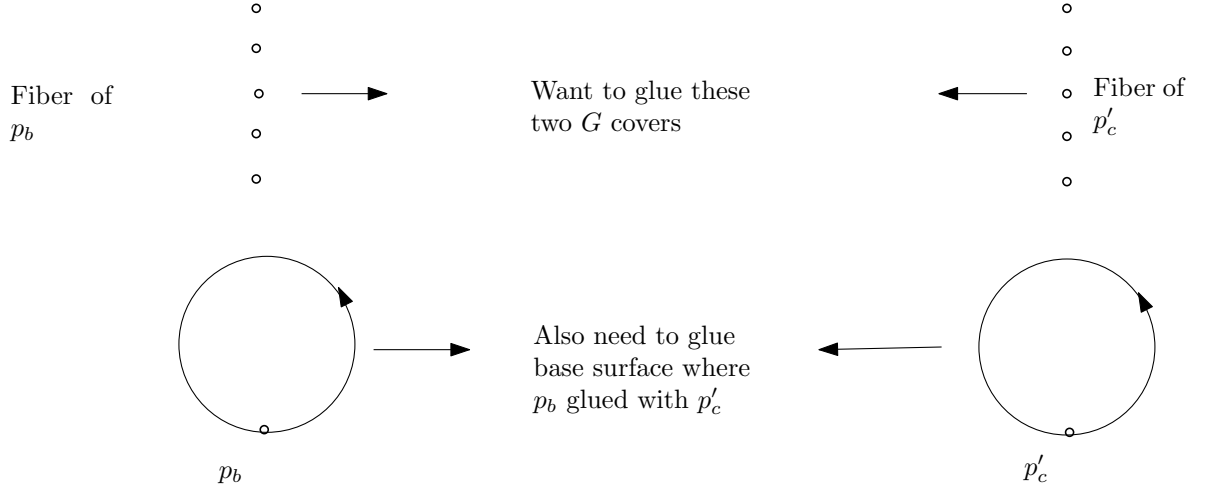
FIGURE 2. Fiber of the two boundary circles we want to glue

Lemma 3. *Let $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ be a G cover of $(\Sigma, \{p_a\})$ and let $(\hat{\pi} : \hat{\Sigma} \longrightarrow \Sigma', \{\hat{p_a}\})$ be a G cover of $(\Sigma', \{p'_a\})$. Take $b \in A(\Sigma)$ and $c \in A(\Sigma')$. We want to glue $(\partial\Sigma)_b$ and $(\partial\Sigma')_c$. Then :*

(1) *If the gluing exists, there is a unique way, up to isomorphism of G covers, to glue these G-covers*
(2) *Gluing is possible iff the monodromy, m_b, of $(\partial\Sigma)_b$ and the monodromy, m_c of $(\partial\Sigma')_c$ are inverse of each other. That is $m_b m_c = 1$.*

Proof. Not only we need to glue $(\partial\Sigma)_b$ and $(\partial\Sigma')_c$ but we also need to glue the cover above it. See the diagram on figure 2. We identify $p_b \in (\partial\Sigma)_b$ with $p'_c \in (\partial\Sigma')_c$ and this will basically tells us how to glue $(\partial\Sigma)_b$ with $(\partial\Sigma')_c$ since, first of all the orientation of the boundary circle comes from the orientation of the surface, and the set of all orientation preserving homeomorphisms from $(\partial\Sigma)_b$ to $(\partial\Sigma')_c$ which map p_b to p'_c is homotopic to each other. This is how we glue $(\partial\Sigma)_b$ with $(\partial\Sigma')_c$. Now what about the cover? Let us take a small neighborhood of p_b and p'_c on the circle $(\partial\Sigma)_b$ and $(\partial\Sigma')_c$ so that the fiber above these neighborhoods of circles break up into disjoint pieces and each piece maps homeomorphically by $\tilde{\pi}$ and $\hat{\pi}$ to these neighborhoods of circle. see the diagram on figure 3
  One of the pieces on the left contains $\tilde{p_b}$ and one of the piece on the right contains $\hat{p_c}$. Also G acts simply transitively on these pieces. By definition, we require the piece containing $\tilde{p_b}$ to be glued with the piece containing $\hat{p_c}$ so that the action of G commutes with the gluing. This fixes how the pieces of G-cover containing $\tilde{p_b}$ will be glued to the pieces of G-cover containing $\hat{p_c}$. Now we move around the circle and repeat the same process until we cover the whole circle, $(\partial\Sigma)_b$ and $(\partial\Sigma')_c$.
    This tells us that there is at most one way to glue. Note that not always we can glue two G- covers. For example take $|G| = 2$. Take one G-cover to be the trivial G-cover of the circle and the other G-cover to be the double cover of the circle. These two G-covers can not be glued.

Each
piece
maps
homeo-
morphi-
cally
by $\tilde{\pi}$ to
the
neigh-
borhood
of p_b

piece contain-
ing
$\tilde{p}_b$

$\tilde{p}_b$

p_b

The neighbor-
hood
of p_b

Each piece
maps   homeo-
morphically
by $\hat{\pi}$ to
the   neighbor-
hood
of p'_c

piece
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$\hat{p}_c$

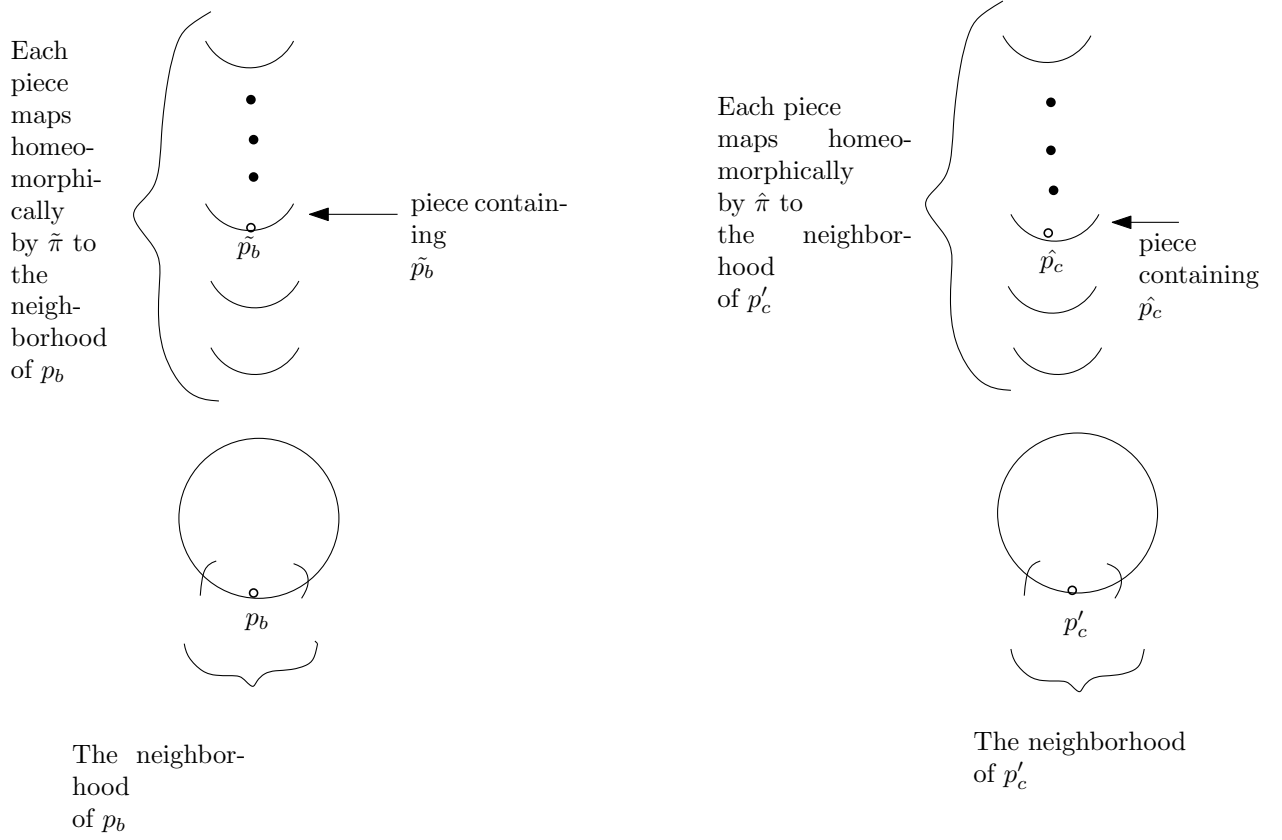$\hat{p}_c$

p'_c

The neighborhood
of p'_c

FIGURE 3. Neighborhood of distinguished points and the fiber
above it

The second part of the lemma is left to the reader. Basically one sees that as we
moved around the whole circle and then glue or we first glue and then move around
the whole circle, in either case we arrived at the same point since the monodromy
are inverse of each other. So we don't have any problem to glue. Readers can
supply the detail.   □

3. STANDARD BLOCK

3.1. Standard sphere, S_n**.** For every $n \geq 0$, we define the standard sphere, S_n,
to be the Reimann sphere $\overline{\mathbb{C}}$ with n disks $|z - k| < \frac{1}{3}$ removed and with the marked
points being $k - \frac{i}{3}$, here $k = 1, 2, 3, ...n$. Of course, we could replace these n
disks with any other n non-overlapping disks with centers on the real line and with
marked points in the lower half plane. Any two such spheres are homeomorphic and
the homeomorphism can be chosen canonically up to homotopy. Note that the set
of boundary components of the standard sphere is naturally indexed by numbers
$1, 2, ...n$. The standard sphere, S_4, with four holes is shown on figure 4

3.2. Standard blocks, $S_n(g_1, ..., g_n; h_1, ..., h_n)$**.** In the paper [BK], where $|G| = 1$,
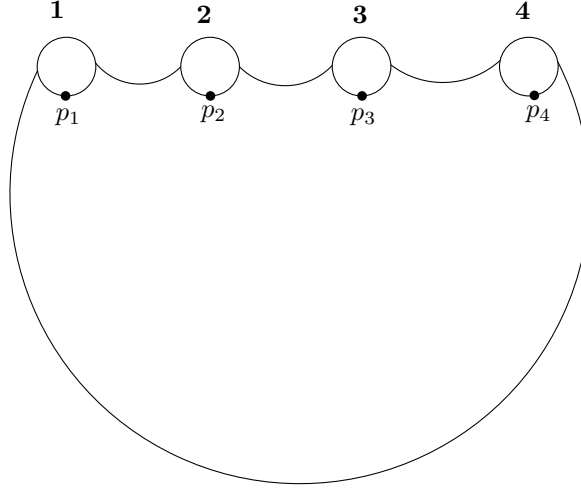standard blocks are just these standard spheres,$S_n, n = 1.2....$ So we need to extend

FIGURE 4. A standard sphere with four holes where the marked points are denoted by $p_1, ... p_4$
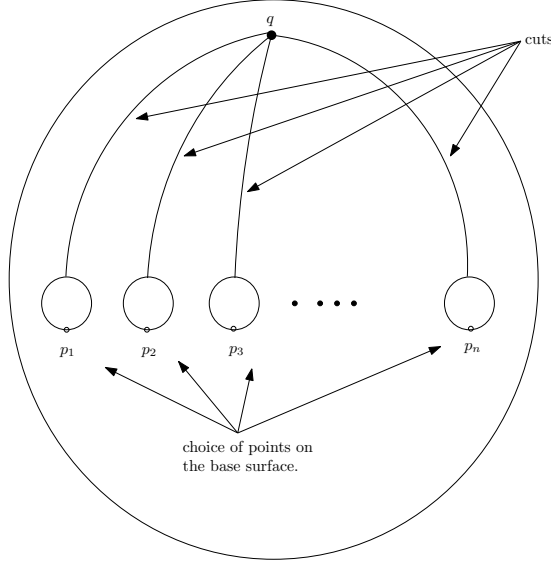

FIGURE 5. Cuts on S_n

this definition to the general case where G is a finite group. To do this we start with the following construction. Let us start with a standard sphere with n holes, S_n and $2n$ elements from G where we denote these $2n$ elements as $\{g_1, ..., g_n\}$ and $\{h_1, ..., h_n\}$ and we also required that $g_1 ... g_n = 1$. First we make the cuts on S_n as in figure 5. Here the point $q \in S_n$ in figure 5 is the point at ∞. In fact q can be chosen to be any point on the upper hemisphere as long as it does not belong to the boundary circles. Then one can easily sees that $S_n \backslash$cuts is simply connected. So G-cover of $S_n \backslash$cuts is trivial. In other word, G-cover of $S_n \backslash$cuts looks like

gluing along i th cut
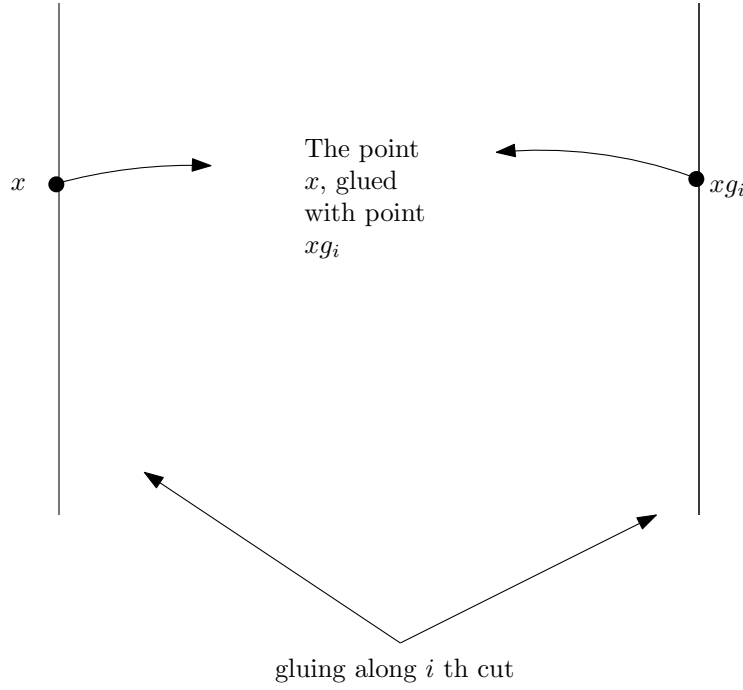
FIGURE 6. A point x on the left hand side of the fiber of the ith cut is glued to xg_i on the right hand side

$$S_n \backslash \text{ cuts } \times G$$
$$\downarrow$$
$$S_n \backslash \text{ cuts}$$

Although there is only one G cover of $S_n\backslash$cuts up to isomorphism, there are a total of $|G|$ many way to identify G cover of $S_n\backslash$cuts with G, but the important thing here to notice is that any such identification is isomorphic. So we chose one such identification. Now consider the ith cut. We want to glue the fiber on the left hand side of the ith cut with the fiber on the right hand side of the ith cut. This identification must preserve the action of the group G. Thus the identification can be given by a multiplication on the right by some element of the group G. We chose this element to be $g_i \in G$. That is we glued along the ith cut by multiplication of the right by g_i. See figure 6. Now we choose $\tilde{p}_i = (p_i, h_i)$ as the choice of point in the fiber of p_i. Thus we end up with a G cover of S_n. The reason we require that $g_1...g_n = 1$ is easy to see. Consider a point (t, x), where $x \in G$, on the G-cover. As we moved along each cut, we multiply on the right by g_i. After moving along all the cut, we end up with (some point on of the base surface , $xg_1g_2...g_n$), but then we must have $x = xg_1g_2...g_n$ or equivalently $g_1...g_n = 1$. Thus from the data $g_1, ..., g_n$ and $h_1, ..., h_n$ where $g_1...g_n = 1$, we have constructed a G cover of S_n.

Definition 7. The G cover of S_n constructed above is called standard block and will be denoted by $S_n(g_1, g_2...., g_n; h_1, h_2...., h_n)$ where we always assume that $g_1 g_2 ... g_n = 1$ although we do not always write this explicitly.

3.3. **When two standard blocks are isomorphic.** Given two G-covers of S_n, we want to know when they are isomorphic, hence the following lemma:

Lemma 4. *Let $S_n(g_1, ..., g_n; h_1, ...h_n)$ and $S_n(g'_1, ..., g'_n; h'_1...h'_n)$ be two standard blocks. Then they are isomorphic iff $\exists x \in G$ so that $x g_i x^{-1} = g'_i$ and $h_i x^{-1} = h'_i$ for $i = 1...n$. We denote the isomorphism, $S_n(\mathbf{g}, \mathbf{h}) \to S_n(\mathbf{g'}, \mathbf{h'})$ by ϕ_x.*

Proof. First assume $S_n(g_1, ..., g_n; h_1, ...h_n)$ and $S_n(g'_1, ..., g'_n; h'_1...h'_n)$ are isomorphic. This means there exist an isomorphism of G-cover which by definition (see section 2.3 on page 4) maps $(p_i, h_i) \longrightarrow (p_i, h'_i)$ and preserve the G-action. So on the fiber, this isomorphism is just the right multiplication by some element $x^{-1} \in G$. Also the following condition must satisfy along the cut:
First glue and then apply isomorphism = First apply isomorphism and then glue. In other word, if t is a point on the fiber above the ith cut, then we must have

$$t g_i x^{-1} = t x^{-1} g'_i$$
$$\text{or } g'_i = x g_i x^{-1} \text{ for } i = 1...n$$

And of course we must have $h_i x^{-1} = h'_i$ from the definition of isomorphism of two G-cover.
The other direction is easier to show. We define the isomorphism on each fiber through the multiplication on the right by x^{-1}. This can be easily seen to satisfy all the required property of isomorphism. $\square$

3.4. **Monodromy of a Standard Block.** The next natural question is to ask what is the monodromy of the standard block, $S_n(g_1, g_2...g_n; h_1, h_2...h_n)$. So we have the following lemma:

Lemma 5. *Let $S_n(g_1, ..., g_n; h_1, ...h_n)$ be a G-cover of S_n. Here we assume the orientation of the boundary circles induces by the outward normal vector according to the right hand rule. See section 2.4 on page 5 for the discussion of how the orientation of the surface induces orientation on the boundary. See the picture on figure 7. Then the monodromy $m_i \in G$ around the ith boundary circle is given by:*
$$m_i = h_i g_i^{-1} h_i^{-1}$$

Proof. Given a parameterization $\alpha : [0,1] \longrightarrow i$th boundary circle of S_n, we lift this path starting from (p_i, h_i), and as we cross the cut labeled by g_i from right to left, we end up with $(*, h_i g_i^{-1})$. See the picture on figure 8.

And as we continue all the way, at the end we arrive at the point $(p_i, h_i g_i^{-1})$. So from the definition of monodromy, the ith monodromy $m_i \in G$ is given by

$$m_i h_i = h_i g_i^{-1} \text{ so}$$
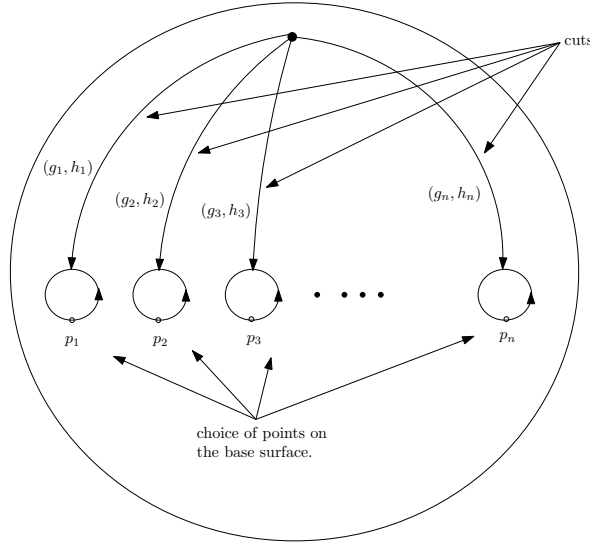$$m_i = h_i g_i^{-1} h_i^{-1}$$

FIGURE 7. observe the orientation of the boundary circles

FIGURE 8.

$\square$
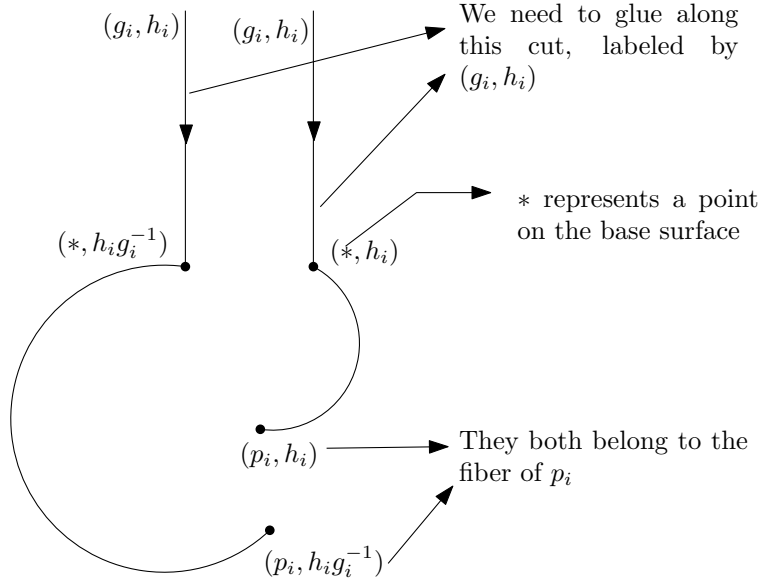
3.5. **Gluing of two standard blocks.** Let the standard block $S_n(g_1, ..., g_n; h_1, ..., h_n)$ and $S_m(u_1, ..., u_m; v_1, ..., v_m)$ are given. We want to know when we can glue these two standard blocks along the i th boundary circle of $S_n(g_1, ..., g_n; h_1, ..., h_n)$ to the j th boundary circle of $S_m(u_1, ..., u_m; v_1, ..., v_m)$. Hence we have the following lemma:

Lemma 6. *i th boundary circle of $S_n(g_1, ..., g_n; h_1, ..., h_n)$ can be glued to the j th boundary circle of $S_m(u_1, ..., u_m; v_1, ..., v_m)$ iff $h_i g_i^{-1} h_i^{-1} = [v_j u_j^{-1} v_j^{-1}]^{-1}$.*

Proof. According to lemma 5 on page 12, monodromy of the i th boundary circle of $S_n(g_1, ..., g_n; h_1, ..., h_n)$ is given by $h_i g_i^{-1} h_i^{-1}$ and the monodromy of the j th boundary circle of $S_m(u_1, ..., u_m; v_1, ..., v_m)$ is given by $v_j u_j^{-1} v_j^{-1}$. Now according to the lemma 3 on page 8, such a gluing exist if and only if these two monodromy are inverse of each other, that is $h_i g_i^{-1} h_i^{-1} = [v_j u_j^{-1} v_j^{-1}]^{-1}$.   □

Remark 3. We will use the notation

$$S_n(g_1, ..., g_n; h_1, ..., h_n) \bigsqcup_{c, h_i, v_j} S_m(u_1, ..., u_m; v_1, ..., v_m)$$

to indicate that the ith boundary of the left standard block is glued along the cut c to the jth boundary of the right standard block.

3.6. **Groupoid.**

Definition 8. A category is called a groupoid if all of its morphism is invertible.

We list two important examples of groupoid which will be needed later.

Example 1. *For a fixed S_n, consider the category, Y_n, defined in the following way:*
obj(Y_n)= G-covers of S_n
mor(Y_n)= isomorphisms of G-covers which are trivial on S_n.
Since only morphisms are isomorphisms, this is obviously a groupoid.

Example 2. *We define the category T_n in the following way:*
obj(T_n) = {standard blocks} = $\{S_n(g_1, g_2, ...g_n; h_1, h_2...h_n)| g_i \in G$ and $h_i \in G$ for $i = 1...n$ and $g_1 g_2...g_n = 1\}$
According to lemma in 3.3 on page 12 all the isomorphism between two standard block can be defined in the following way:
mor($S_n(g_1, g_2, ...g_n; h_1, h_2...h_n), S_n(g_1', g_2', ...g_n'; h_1', h_2'...h_n')$)=$\{x \in G| x g_i x^{-1} = g_i'$ and $h_i x^{-1} = h_i'$ for $i = 1...n\}$. Since given a morphism $x \in G$ in this category the inverse morphism is $x^{-1} \in G$, every morphism is invertible. So the category T_n is in fact a groupoid.

We have the following important lemma:

Lemma 7. *The groupoid Y_n is equivalent to the groupoid T_n. See example 1 and 2 above for the description of Y_n and T_n.*

Proof. To prove this lemma, we will use a well known theorem from category theory.

Theorem 1. *Let A and B be two categories, and $F : A \to B$ is a covariant functor, so that the following two conditions hold:*

  (1) *For any two objects, $X, Y \in Obj(A)$, the map $Mor_A(X, Y) \longrightarrow Mor_B(F(X), F(Y))$, induced by F, is bijective.*

(2) *The map F is essentially surjective. That is given any object, $V \in Obj(B)$, there exist an object, $U \in Obj(A)$ so that $F(U) \cong V$. Here the symbol $\cong$ means isomorphic.*

Then F is in fact an equivalence of categories.

Proof. For a proof of this theorem, see any standard book on category theory, for example "Categories for the working Mathematician" by Mac Lane, see [C]. □

Now let us come back to the proof of our lemma. We will construct a functor $V : T_n \rightarrow Y_n$ so that V satisfies the two conditions of Theorem 1. Then this V will define the equivalence between T_n and Y_n. Given a standard block, $S_n(g_1, g_2, ...g_n; h_1, h_2...h_n)$ where $g_i \in G$ and $h_i \in G$ for $i = 1...n$ and $g_1 g_2 ... g_n = 1$, this standard block, in particular, a G cover of S_n. That is

$V(S_n(g_1, g_2, ...g_n; h_1, h_2...h_n)) = $ the G cover of S_n corresponding to the standard block

Similarly, if $\phi_x : S_n(g_1, g_2, ...g_n; h_1, h_2...h_n) \rightarrow S_n(g_1', g_2', ...g_n'; h_1', h_2'...h_n')$ is a morphism between two standard blocks, then $V(\phi_x)$ is the same morphism between the G covers.

To show, this functor V satisfies the condition 1 of the above theorem, we just refer to the lemma 4 on page 12. First note that, by this lemma, given any two standard blocks, either there exist an unique morphism, ϕ_x, between them or there is no morphism between them. In either case, the condition 1 is obviously satisfied.

For condition 2, let $\pi : \tilde{S}_n \rightarrow S_n$ be a G-cover of S_n. First, we make the same cuts on S_n as figure 5 on page 10. Then the G-cover of $S_n \backslash$cuts is trivial. In other word, the G-cover of $S_n \backslash$cuts can be identified with

$$S_n \backslash \text{ cuts } \times G$$

$$\downarrow$$

$$S_n \backslash \text{ cuts}$$

Now we need to identify the components of $S_n \backslash$ cuts $\times G$ with the group G. Here we have choice. So we make some choice, it does not matter how we want to do this. Now we basically repeat the same construction when we describe standard blocks, namely, from this trivial cover, to get the original cover we started with, we need to glue the cover along the cuts. See the picture on figure 9. This gluing must preserve the action of G on the fiber. So to glue along the i-th cut, there must exist $g_i \in G$ so that the point t on one side of the cut must glued with the point tg_i from the other side of the cut. Thus for a total of n-cut we get $\{g_1, g_2...g_n \in G\}$. Also to start with, our G-cover comes with a point (p_i, h_i) on the fiber above p_i. This gives us $\{h_1, h_2...h_n \in G\}$. To show $g_1...g_n = 1$ is easy. Consider a point (t, x), where $x \in G$, on the G-cover. As we moved along each cut, we multiply on the right by g_i. After moving along all the cut, we end up with (some point on of the base surface ,$xg_1 g_2...g_n$), but then we must have $x = xg_1 g_2...g_n$ or equivalently $g_1...g_n = 1$. Thus the G cover $\tilde{S}_n$ is isomorphic to the standard block $S_n(g_1, ...g_n; h_1, ...h_n)$. This shows that the functor V satisfies the second requirement of the above theorem. □

The point
x, glued
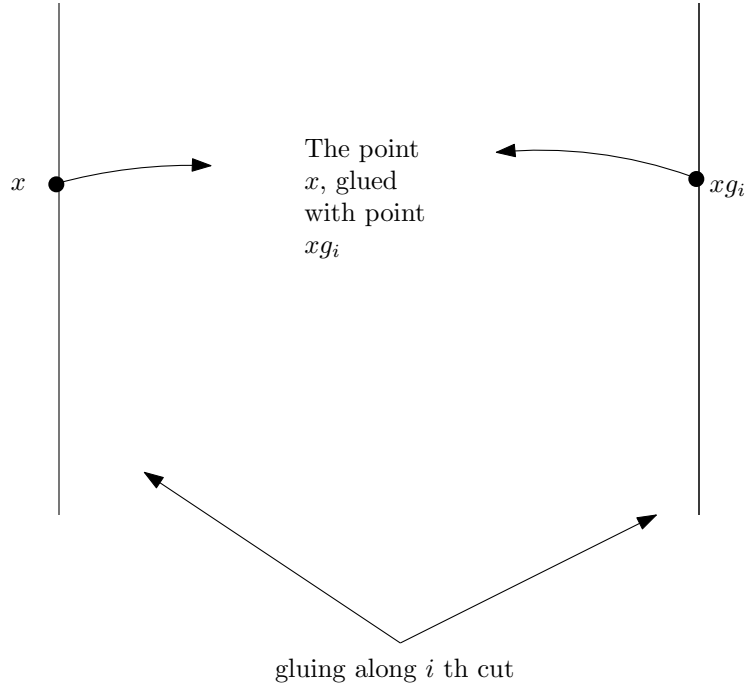with point
xg_i

x

xg_i

gluing along i th cut

FIGURE 9.

Remark 4. In particular, the above theorem shows that every G cover of S_n is isomorphic to a standard block.

3.7. **Review of the parameterization for the case** $|G| = 1$**.** For the readers convenience, we will review the concept of parameterization for the trivial case, $|G| = 1$, from [BK]. We begin with some definitions.

Definition 9. Let Σ be an extended surface. A cut system, C, on Σ is a finite collection of smooth, simple closed non-intersecting curves on Σ such that each connected component of the complement, $\Sigma \backslash C$, is a surface of genus zero. In this paper, we will always assume the surface Σ has genus 0. So in this case, the requirement that each connected component of the complement, $\Sigma \backslash C$, is a surface of genus zero is always satisfied.
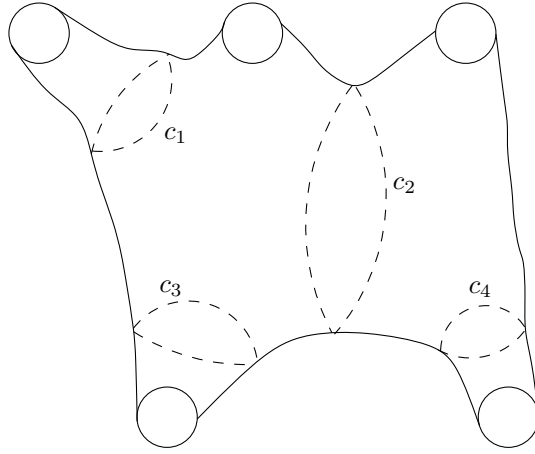
An example of a cut system is given in figure 10.

Definition 10. Let Σ be an extended surface with genus zero. A parameterization without cuts of Σ is a homotopy equivalence class of homeomorphisms $\phi : \Sigma \simeq S_n$, where S_n is the standard sphere with n holes.

Definition 11. Let Σ be a an extended surface. A parameterization, P, of Σ is a collection $(C, \{\phi_a\})$, where C is a cut system on Σ and ϕ_a are parameterization without cuts of the connected components Σ_a of $\Sigma \backslash C$, i.e. homotopy equivalence class of homeomorphisms $\phi_a : \Sigma_a \simeq S_{n_a}$.

Definition 12. Let S_n be the standard sphere with n holes. We let m_0 be the graph on it, shown in figure 11, for $n = 5$. This graph has a distinguished edge- one

Here the cut system, $C = \{c_1, c_2, c_3, c_4\}$
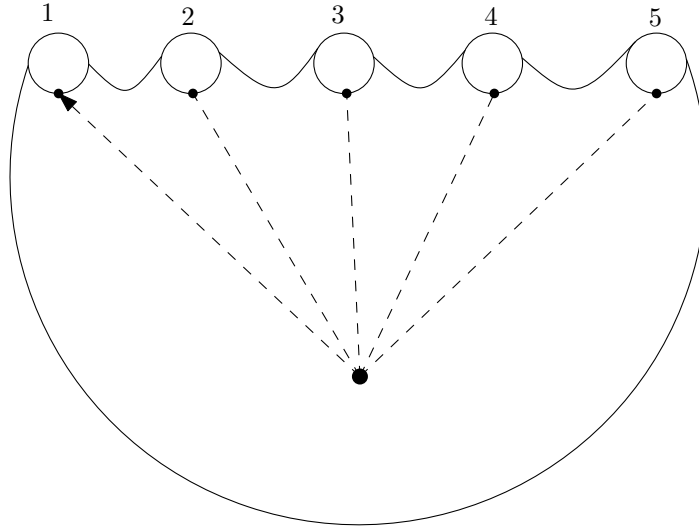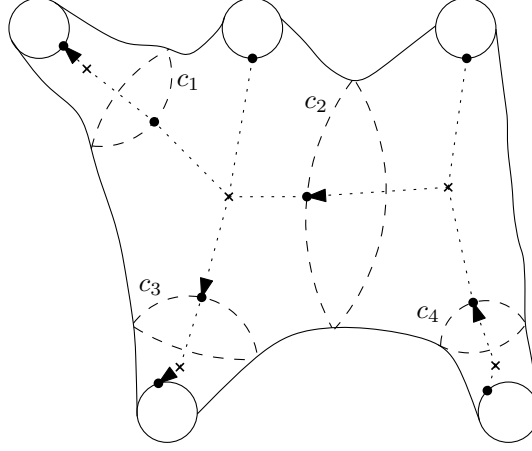
FIGURE 10. An example of a cut system

FIGURE 11. Standard marking on S_5, sphere with 5 holes

which connected to the boundary component labeled by 1. This distinguished edge has been marked by an arrow. We call m_0, the standard marking without cuts on S_n.

Let Σ be an extended surface with genus zero. A marking without cuts of Σ is a graph, m, on Σ with one distinguish or marked edge such that $m = \phi^{-1}(m_0)$ for some homeomorphism $\phi : \Sigma \to S_n$. The graphs are considered up to isotopy of Σ.

Definition 13. Let Σ be an extended surface. A marking, M of Σ is a pair, (C, m), where C is a cut system on Σ and m is a graph on Σ with some distinguished edges

Here the cut system, $C = \{c_1, c_2, c_3, c_4\}$ and the
marking M is as shown on the figure.

FIGURE 12. An example of a marked surface
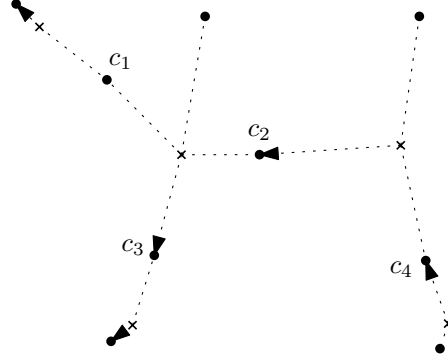
Here the cut system, $C = \{c_1, c_2, c_3, c_4\}$ and they
are placed on the vertices corresponding to the
cut.

FIGURE 13. Marked graph without the surface

such that it gives a marking without cuts on each of the connected component of
$\Sigma \backslash C$. We will denote the set of all marking of a surface Σ modulo isotopy by $M(\Sigma)$.
A marked surface is an extended surface, Σ, together with a marking, M on it.

An example of a marked surface is shown on figure 12. Sometimes, for the
convenience of drawing, we drop the surface from the picture and just draw the
marked graph if no confusion arise. So for example, the marked surface on figure
12 may be just drawn as in figure 13.

3.8. **Parameterization in the general case.** In this paper we will only consider G cover of extended surface for which the base surface is connected and has genus zero. But this definition makes sense in the general case also.

Definition 14. Let $\tilde{\Sigma} \to \Sigma$ be a G cover of an extended surface, Σ. A parameterization of $\tilde{\Sigma}$ is an isomorphism of this G-cover with one or gluing of several standard blocks. We have defined what we meant by gluing of G-cover (see section 2.6.1 on page 7). That is if f is a parameterization of the G-cover $\tilde{\Sigma}$ then f is just an isomorphism which looks like :

$$f : \tilde{\Sigma} \longrightarrow S_{n_1}(g_1^1, ...g_{n_1}^1; h_1^1, ...h_{n_1}^1) \bigsqcup_{c_i, h_p^1, h_q^2} ..... \bigsqcup_{c_j, h_t^{k-1}, h_l^k} S_{n_k}(g_1^k, ...g_{n_k}^k; h_1^k, ...h_{n_k}^k)$$

For the explanation of the above notation see the remark on section 3.5 on page 13.

3.9. **Visualizing parameterization.** We need some kind of graphical way to represent parameterization of G-cover of extended surface just as in the case of $|G| = 1$ which is described in the subsection 3.7 on page 16. So we need a similar kind of machinery in this general case. The lemma 7 gives us a way to visualize G-covers of extended surface. Given a G-cover of an extended surface,$\tilde{\Sigma} \to \Sigma$, first we marked the base surface, Σ. In other word, we identify the base surface, Σ, with one or gluing of several standard spheres. For details about the marking of an extended surface see the subsection 3.7 on page 16. This marked base surface gives the parameterization of the base surface, Σ, with one or gluing of several S_n where n may vary. Let C be the cut system of this marking. If we restrict the G cover, $\tilde{\Sigma}$, to each connected component of $\Sigma \backslash C$, then the whole G cover can be realized as a gluing of all this restriction. Each such restriction is isomorphic to one of the standard block, $S_n(g_1, ..., g_n; h_1, ..., h_n)$, where n may vary for each restriction. So the whole G cover, $\tilde{\Sigma} \to \Sigma$, can be identify with the gluing of several standard blocks (one for each restriction). We can include all this data into the surface, Σ as follows:
We label each edge of our marking graph with a pair (g_i, h_i) which comes from the identification of the restriction of $\tilde{\Sigma}$ to the component containing the edge. See the proof of the lemma in section 7 to see how to assign (g_i, h_i) to each edge. See a typical picture on figure 14. sometimes, we drop the picture of the surface and just draw the graph for simplicity, if no confusion arise. So for example, we will usually draw the picture on figure 14 as a simple figure in 15

This gives a visual presentation of the parameterization of our G-cover of extended surface with one or gluing of several standard blocks.

4. Moves

At this point, we want to remind our readers about our main goal of this paper. we will eventually define a 2-dimensional CW complex $M(\tilde{\Sigma}, \Sigma)$, which has the set of all parameterization of $\tilde{\Sigma}$ as the set of vertices. The edges of $M(\tilde{\Sigma}, \Sigma)$ will be directed; we call them moves. It is convenient to look at $M(\tilde{\Sigma}, \Sigma)$ as a groupoid with objects—all vertices and morphisms between two vertices—the set of homotopy classes of paths on the edges of $M(\tilde{\Sigma}, \Sigma)$ from the first vertex to the second one (going along an edge in the direction opposite to its orientation is allowed). We will use group notation writing a path composed of edges $E_1, E_2, \ldots$

glued along the cut c. We must have $yx^{-1}y^{-1} = [wz^{-1}w^{-1}]^{-1}$ for this gluing to exist. See lemma 6 for detail.
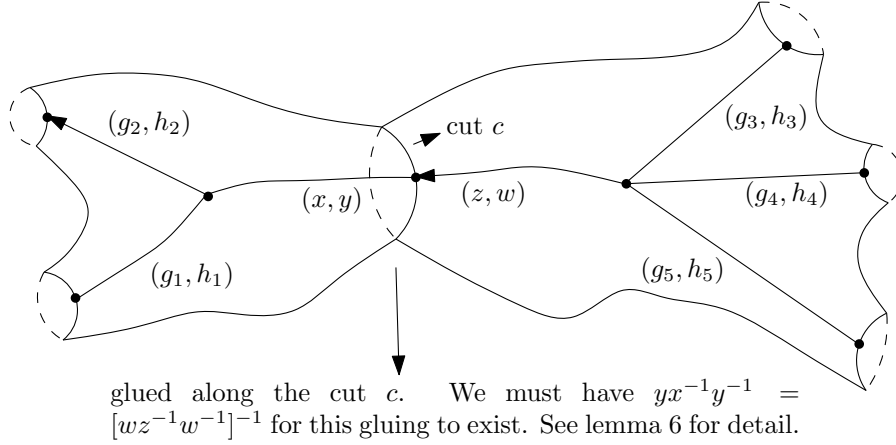
FIGURE 14. Visualization of the parameterization of G-cover with the gluing of $S_3(g_1, g_2, x; h_1, h_2, y)$ and $S_4(x^{-1}, g_3, g_4, g_5; y, h_3, h_4, h_5)$
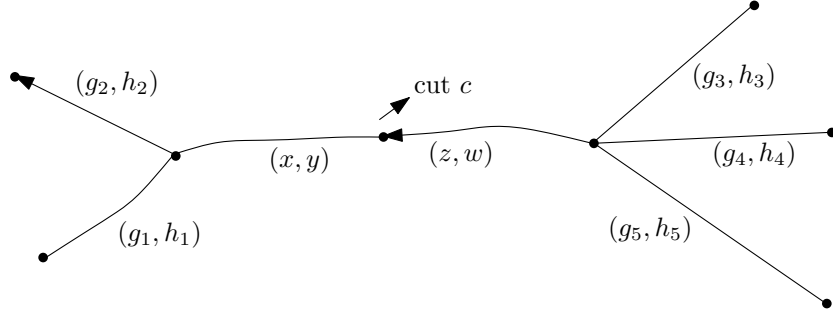

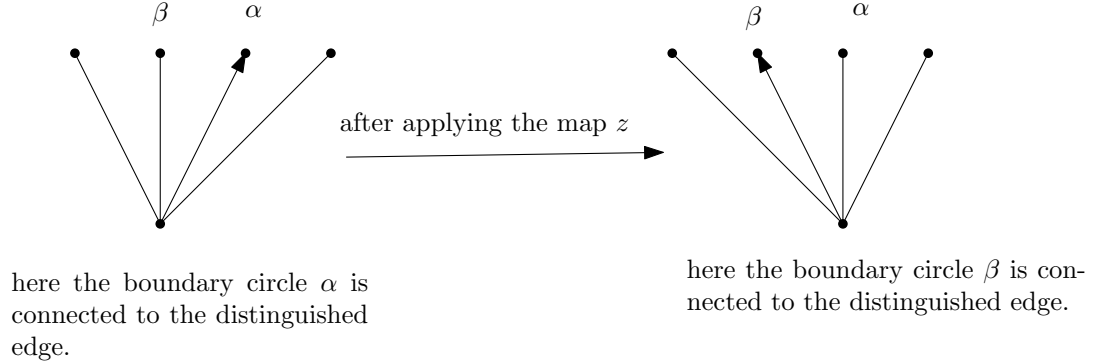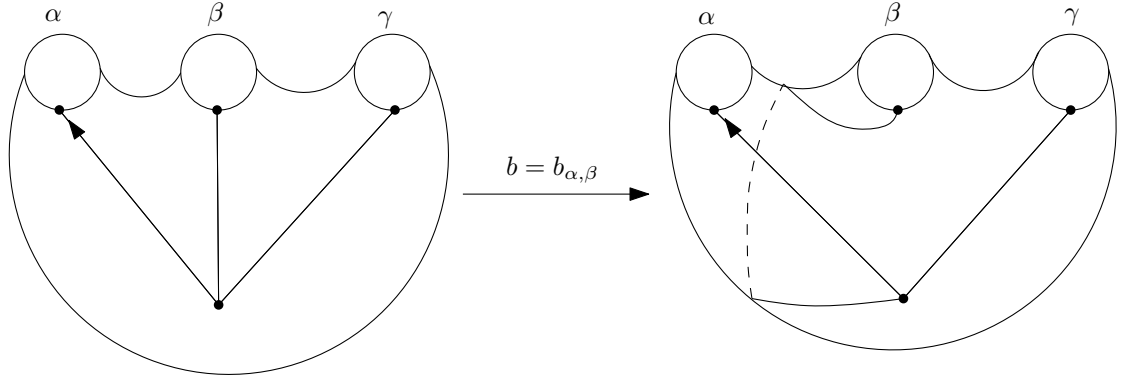FIGURE 15. We usually do not draw surface and just draw the graph

as a product $E_1 \ldots E_2$, and we will write E^{-1} if the edge E is traveled in the opposite direction. Then the 2-cells are interpreted as relations among the moves: we will write $E_1 \cdots E_k = id$ if the closed loop formed by the edges $E_1, \ldots, E_k$ is contractible in $M(\tilde{\Sigma}, \Sigma)$; if we want to specify the base point for the loop, we will write $E_1 \cdots E_k(M) = id(M)$. We will write $E : M \to M'$ if the edge E goes from M to M'.

Our Main Theorems state that the complex $M(\tilde{\Sigma}, \Sigma)$ is connected and simply-connected. This main theorem will be described in detail after we describe the moves or edges in this section.

4.1. **Standard Morphism.** Before describing our moves, we need to define some standard morphisms between our standard blocks. These are described in terms of lifting of certain morphisms between base surfaces S_n. These are:

$$z : S_n \longrightarrow S_n$$
$$b : S_3 \longrightarrow S_3$$

here the boundary circle α is connected to the distinguished edge.

here the boundary circle β is connected to the distinguished edge.

FIGURE 16. A pictorial description of z map for $n = 4$

FIGURE 17. Visual description of b map

$$\alpha_{k,l} : S_{k+1} \sqcup S_{l+1} \longrightarrow S_{k+l}$$

For a more elaborate description of these morphism, see the paper [BK]. Here we will just give a quick description.

1: z is the rotation of the sphere which cyclically permutes the boundary circles. That is if m is a marking on S_n then $z(m)$ will be the same marking (of course up to homotopy) on S_n but with a different distinguished edge. See the figure 16 where $n = 4$.

2: Now we will describe briefly the braiding, b, see [BK] for detail. If we label the boundary circles of S_3 by α, β, γ, then sometimes we will denote by $b_{\alpha,\beta}$ the braiding of the α and β component of S_3. Let m is the graph on S_3 shown on the left hand side of figure 17. Then we define the $b_{\alpha,\beta}$ by figure 17.

3: $\alpha_{k,l}$ is the identification of the result of gluing S_{k+1} and S_{l+1} (along the $(k+1)$ th boundary component of the first one with the 1 st boundary component of the last one) with, S_{k+l}, the standard sphere with $(k+l)$ hole. For more detail description see [BK].

Lemma 8. *Each of the above defined morphisms between S_n ($n = 3$ in the case of b) can be uniquely lifted to the G-cover of S_n, $S_n(g_1, ...g_n; h_1, ...h_n)$, described by:*

$$\tilde{z} : S_n(g_1...g_n; h_1...h_n) \longrightarrow S_n(g_n, g_1...g_{n-1}; h_n, h_1...h_{n-1})$$
$$\tilde{b} : S_3(g_1, g_2, g_3; h_1, h_2, h_3) \longrightarrow S_3(g_1 g_2 g_1^{-1}, g_1, g_3; h_2 g_1^{-1}, h_1, h_3)$$
$$\tilde{\alpha_{k,l}} : S_{k+1}(\mathbf{g}, \mathbf{h}) \sqcup S_{l+1}(\mathbf{g'}, \mathbf{h'}) \longrightarrow S_{k+l}(\mathbf{g''}, \mathbf{h''})$$

where

$$\begin{aligned}
\mathbf{g} &= (g_1...g_{k+1}) \\
\mathbf{h} &= (h_1...h_{k+1}) \\
\mathbf{g'} &= (g'_1, ...g'_{l+1}) \\
\mathbf{h'} &= (h'_1, ...h'_{l+1})
\end{aligned}$$

So that $h_{k+1} g_{k+1}^{-1} h_{k+1}^{-1} = [h'_1 g'^{-1}_1 h'^{-1}_1]^{-1}; h_{k+1} = h'_1$
And

$$\begin{aligned}
\mathbf{g''} &= (g_1...g_k, g'_2...g'_{l+1}) \\
\mathbf{h''} &= (h_1...h_k, h'_2...h'_{l+1})
\end{aligned}$$

Remark 5. Notice that we need $h_{k+1} g_{k+1}^{-1} h_{k+1}^{-1} = [h'_1 g'^{-1}_1 h'^{-1}_1]^{-1}$ so that it is possible to glue the $(k+1)$ th boundary circle of S_{k+1} with 1st boundary circle of S_{l+1}. This is to make sure that the monodromy (see 2.6.1 on page 7) of $(k+1)$ th boundary circle of S_{k+1} is inverse of the monodromy of 1st boundary circle of S_{l+1} (recall that the orientation of two boundary circles must be opposite for glue to exist). The second equality, $h_{k+1} = h'_1$, is an extra one which is not needed for the gluing to exist but we assume this extra condition whenever we want to apply $\tilde{\alpha_{k,l}}$.

Proof. First consider the G-covers,$S_n(g_1...g_n; h_1...h_n)$ and $S_n(g_n, g_1...g_{n-1}; h_n, h_1...h_{n-1})$, and the map $z : S_n \longrightarrow S_n$. Our base surface, S_n has genus zero and the map z preserves the monodromy which can be easily checked by hand. So according to lemma 2 (see 2 on page 6), this map z can be uniquely lifted to the G-cover. Similarly, in all the other cases, all we have to check that the maps $b : S_3 \longrightarrow S_3$ and $\alpha_{k,l} : S_{k+1} \sqcup S_{l+1} \longrightarrow S_{k+l}$ preserve the monodromy. This can be easily checked by hand, using the lemma 6. $\qquad\square$

4.1.1. *The standard morphism,* $\phi_x, x \in G$. This is just a reminder to the reader. We have already defined ϕ_x move before. This is the morphism between standard blocks defined by
$\phi_x : S_n(g_1...g_n; h_1...h_n) \longrightarrow S_n(x g_1 x^{-1}...x g_n x^{-1}; h_1 x^{-1}...h_n x^{-1})$. See lemma 4 on page 12.

4.2. **Z, B ,and F Move.** For $|G| = 1$, these moves are described in the paper [BK]. Reader might want to read this first before continue. We again want to remind our readers that our base surface will always have genus 0, unless otherwise specified. We will first have three moves similar to the Z, B and F moves of the paper [BK]. These moves will be called **Z**-move, **B**-move and **F**-move respectively.

Note that we use boldface letter to denote these three moves so that it does not get confused with the Z, B, and F moves of the paper [BK]. Then we will have two extra moves which do not have any correspondence to the paper [BK]. So again Z, B and F will denote the moves in the case of $|G| = 1$ and $\mathbf{Z}$, $\mathbf{B}$,and $\mathbf{F}$ will denote the moves in the general case. Each move will take a parameterization to another parameterization. So let us start describing these moves in more detail:

4.2.1. **Z move**. Given a G-cover $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ of $(\Sigma, \{p_a\})$, and a parameterization of this G-cover with a standard block. This means we have an isomorphism, f, of this G-cover with one of the standard block say $S_n(g_1...g_n; h_1...h_n)$. **Z** move takes this parameterization f to the parameterization $\tilde{z} \circ f$ where $\tilde{z}$ is the standard morphism defined in the lemma 5. This new parameterization identify the original G-cover with $S_n(g_n, g_1...g_{n-1}; h_n, h_1...h_{n-1})$. Look at the diagram below

$$\underbrace{\text{Original G-cover} \xrightarrow{f} S_n(g_1...g_n; h_1...h_n) \xrightarrow{\tilde{z}} S_n(g_n, g_1...g_{n-1}; h_n, h_1...h_{n-1})}_{\tilde{z} \circ f = \mathbf{Z}(f), \ \mathbf{Z} \text{ move applied to f}}$$

4.2.2. **B move**. Given a G-cover $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ of $(\Sigma, \{p_a\})$, and a parameterization of this G-cover with a standard block, $S_3(g_1, g_2, g_3; h_1, h_2, h_3)$. This means we have an isomorphism,f, of this G-cover with $S_3(g_1, g_2, g_3; h_1, h_2, h_3)$. **B** move takes this parameterization f to the parameterization $\tilde{b} \circ f$ where $\tilde{b}$ is the standard morphism defined in the lemma 8. This new parameterization identify the original G-cover with $S_3(g_1 g_2 g_1^{-1}, g_1, g_3; h_2 g_1^{-1}, h_1, h_3)$.

4.2.3. **F move**. Here the setup is a little different from the above two moves. Fist, let Σ be an extended surface of genus 0 with one cut $\{c\}$. This cut $\{c\}$ divides Σ into two pieces. Say, G-cover of one piece is parameterized with $S_{k+1}(\mathbf{g}, \mathbf{h})$ and the other with $S_{l+1}(\mathbf{g}', \mathbf{h}')$. That is the parameterization, f, is given by:

$$f : \tilde{\Sigma} \longrightarrow S_{k+1}(\mathbf{g}, \mathbf{h}) \bigsqcup_{c, h_{k+1}} S_{l+1}(\mathbf{g}', \mathbf{h}')$$

where

$$\begin{aligned}
\mathbf{g} &= (g_1...g_{k+1}) \\
\mathbf{h} &= (h_1...h_{k+1}) \\
\mathbf{g}' &= (g_1', ...g_{l+1}') \\
\mathbf{h}' &= (h_1', ...h_{l+1}')
\end{aligned}$$

And also $h_{k+1} g_{k+1}^{-1} h_{k+1}^{-1} = [h_1' g_1'^{-1} h_1'^{-1}]^{-1}; h_{k+1} = h_1'$
Thus the parameterization of the whole G-cover, f, identifies the G-cover with the result of gluing these two standard blocks along the $k + 1$ st boundary circle of one side with the 1st boundary circle of the other. See the picture on figure 18 for more detail. Now by applying **F** move to f, we get a new parameterization $\tilde{\alpha_{k.l}} \circ f$. Note that, $\tilde{\alpha_{k.l}} \circ f$ is a parameterization of G-cover of Σ with no cuts with $S_{k+l}(\mathbf{g}'', \mathbf{h}'')$
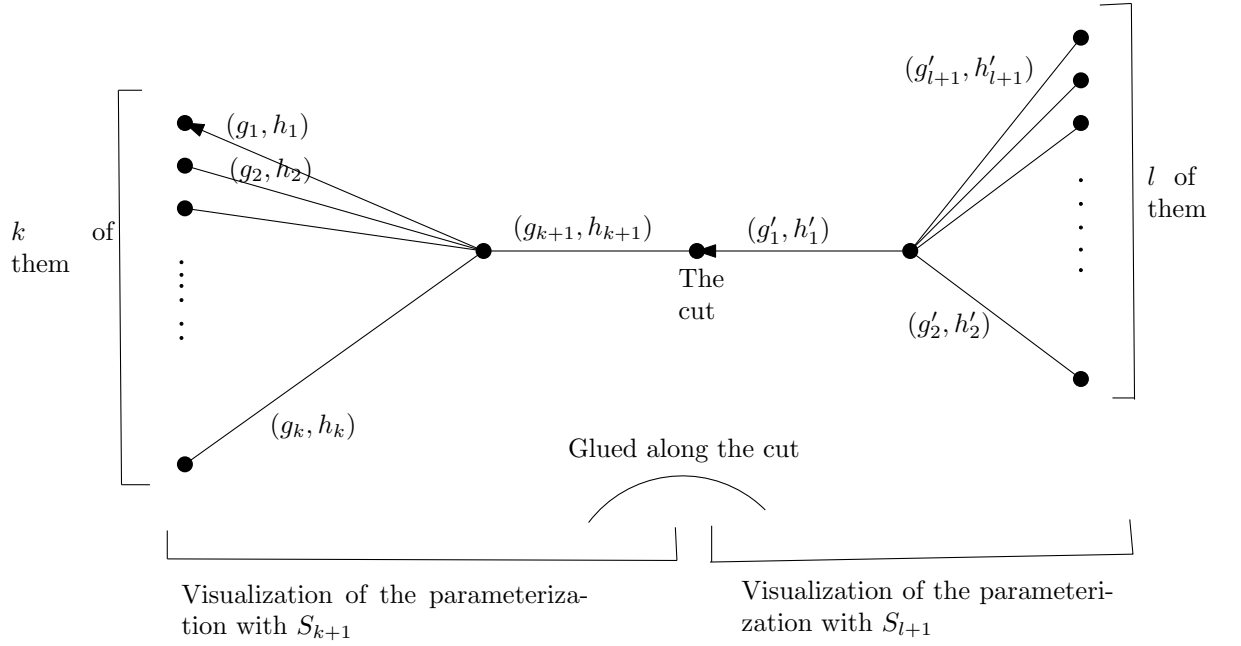
FIGURE 18. Visual description of the parameterization of f

where

$$\begin{aligned} \mathbf{g}'' &= (g_1...g_k, g_2'...g_{l+1}') \\ \mathbf{h}'' &= (h_1...h_k, h_2'...h_{l+1}'). \end{aligned}$$

See the figure 19.

Remark 6. If two components are glued along a cut where the edges corresponding to this cut are labeled by (x, y) and (z, w), then for this gluing to exist, we must have

$$xy^{-1}x^{-1} = [zw^{-1}z^{-1}]^{-1}$$

See subsection 3.5 on page 13 for more detail explanation. Now to apply **F** move to this cut, we also required the extra condition that $y = w$. This will imply that $xz = 1$. The requirement, $y = w$, is not necessary for gluing to exist but we only require this whenever we want to apply **F** move to a cut.

A remark about notation: Note that by applying a **F** move, we are removing a cut. If the cut for one component is labeled by (x, y) then the cut for the other component must be labeled by (x^{-1}, y) for **F** move to apply. See the picture on figure 20. This $x \in G$ is completely determined by the label of all the other cuts since they all must multiplied to $1 \in G$. But for the second component y, we have a choice. So it is better to encode the cut c and the label y in the notation. From now on, a **F** move will always be denoted by $\mathbf{F_{c,y}}$ to emphasize the cut, c, and the choice of point, y.
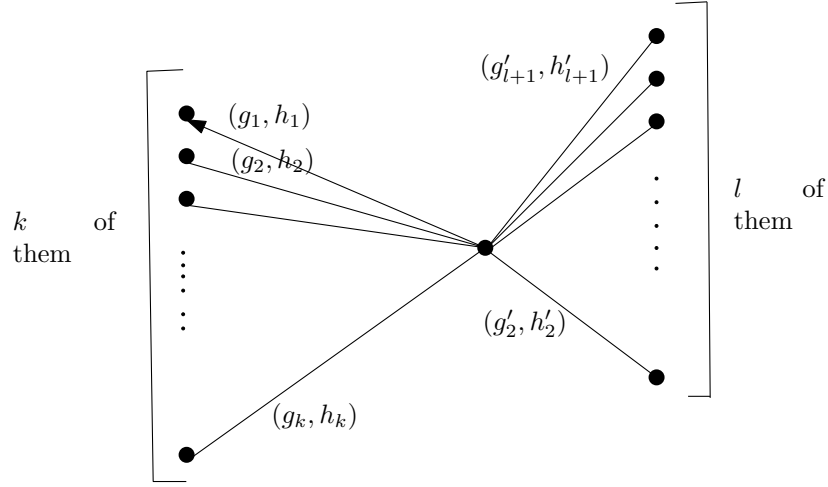
The cut is gone when we apply **F** move to the parameterization f

FIGURE 19. Visualization of the parameterization $\mathbf{F}(f)$

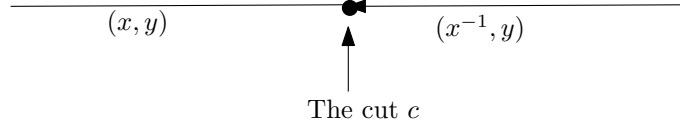FIGURE 20.

4.3. $\mathbf{P_x}$-move where $x \in G$. Beside **Z**, **B** and **F** move, we need one more move which we will denote by $\mathbf{P_x}$ where $x \in G$. This $\mathbf{P_x}$-move is defined in the following way:

Given a G-cover $(\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p_a}\})$ of $(\Sigma, \{p_a\})$, and a parameterization of this G-cover with a standard block. This means we have an isomorphism, f, of this G-cover with one of the standard block say $S_n(g_1...g_n; h_1...h_n)$. $\mathbf{P_x}$ move takes this parameterization f to the parameterization $\phi_x \circ f$ where ϕ_x is the morphism defined by

$\phi_x : S_n(g_1...g_n; h_1...h_n) \longrightarrow S_n(xg_1x^{-1}...xg_nx^{-1}; h_1x^{-1}...h_nx^{-1})$. See the lemma 4 for the description of $\phi(x)$. See the diagram below:

$$\tilde{\Sigma} \xrightarrow{f} S_n(g_1....g_n; h_1...h_n) \xrightarrow{\phi_x} S_n(xg_1x^{-1}...xg_nx^{-1}; h_1x^{-1}...h_nx^{-1})$$

$$\phi_x \circ f = \mathbf{P}_x(f)$$

4.4. $\mathbf{T^c_{z,y}}$ **move.** Say we have the following parameterization, f, of our G-cover:

$$f : \tilde{\Sigma} \longrightarrow S_n(g_1...g_{n-1}, x; h_1...h_{n-1}, y) \bigsqcup_{c,y} S_m(x^{-1}, u_1...u_{m-1}; y, v_1...v_{m-1})$$
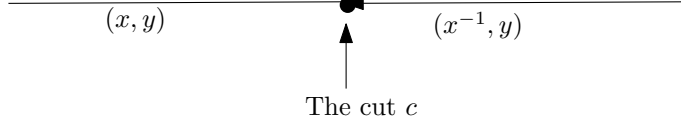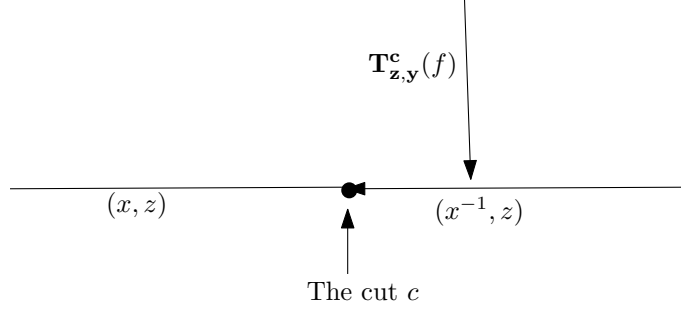
FIGURE 21. The sequence of diagram describing **T** move

c is the cut where we glue the two components. Note that all conditions are satisfied so that the gluing make sense. Here x is determine by all the other element, namely, $x = (g_1...g_{n-1})^{-1} = (u_1...u_{m-1})^{-1}$, but y is not determine; we can choose y freely. By definition, $\mathbf{T}^{\mathbf{c_i}}_{\mathbf{z,y}}(f)$ is the following parameterization:

$$\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}}(f) : \tilde{\Sigma} \to S_n(g_1..g_{n-1}, x; h_1..h_{n-1}, z) \bigsqcup_{c,y} S_m(x^{-1}, u_1..u_{m-1}; z, v_1..v_{m-1})$$

Where we replace the choice of "y" with the choice of "z".

Remark 7. Although we introduce this new move, $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}}$, this can in fact be thought of as composition of two **F** move. Namely $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}} = \mathbf{F}^{-1}_{\mathbf{c,z}}\mathbf{F}_{\mathbf{c,y}}$. The move, $\mathbf{F}_{\mathbf{c,y}}$, will remove the cut, c, with the choice of point y, while the move $\mathbf{F}^{-1}_{\mathbf{c,z}}$ will replace the cut, c, but this time with the choice of point z. The reason for introducing such an extra move will be clear later, but introducing this new move $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}}$ and adding the relation $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}} = \mathbf{F}^{-1}_{\mathbf{c,z}}\mathbf{F}_{\mathbf{c,y}}$ will not do any harm, since one can easily sees that the complex $M(\tilde{\Sigma}, \Sigma)$ without the $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}}$ move and $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}} = \mathbf{F}^{-1}_{\mathbf{c,z}}\mathbf{F}_{\mathbf{c,y}}$ relation is connected and simply-connected iff the complex $M(\tilde{\Sigma}, \Sigma)$ with the $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}}$ move and $\mathbf{T}^{\mathbf{c}}_{\mathbf{z,y}} = \mathbf{F}^{-1}_{\mathbf{c,z}}\mathbf{F}_{\mathbf{c,y}}$ relation is connected and simply-connected.

See the picture on figure 21 for a visual description of **T** move.

4.5. **Moves, when more than one standard blocks are glued together.** Let $\tilde{\Sigma}$ be a G cover of Σ. Also let f be a parameterization of this G cover with gluing
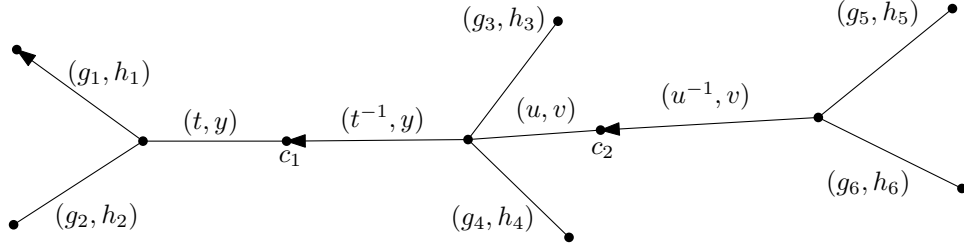
Parameterization f, where f identify the G cover with a gluing of three standrd blocks. We denote the two cuts by c_1 and c_2. Also we assume that all the condition for gluing satisfy.

$$\mathbf{id} \sqcup \mathbf{P_x} \sqcup \mathbf{id}(f)$$
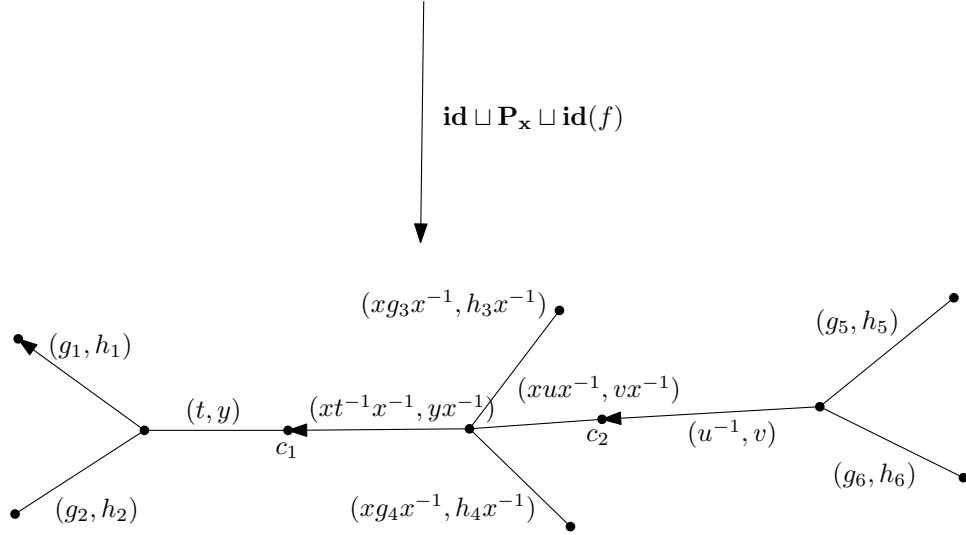
Note that we apply the $\mathbf{P_x}$ move to the middle component leaving the other two component unchanged

FIGURE 22. An example of a move when having more than one component.

of several standard blocks. In other word,

$$f : \tilde{\Sigma} \to S_{n_1}(\mathbf{g_1}, \mathbf{h_1}) \underbrace{\bigsqcup}_{glued} S_{n_2}(\mathbf{g_2}, \mathbf{h_2}) \underbrace{\bigsqcup}_{glued} ..... \underbrace{\bigsqcup}_{glued} S_{n_k}(\mathbf{g_k}, \mathbf{h_k})$$

This parameterization of f, by the restriction on each component, can be realized as a gluing of k parameterization. That is $f = f_1 \sqcup f_2 \sqcup ..... \sqcup f_k$. Let $\mathbf{E}$ be one of the above five moves, that is $\mathbf{E} \in \mathbf{Z}, \mathbf{B}, \mathbf{F}, \mathbf{P_x}, \mathbf{T}$. Then by the move $id \sqcup id \sqcup .... \sqcup \mathbf{E} \sqcup ... \sqcup id(f)$, where $\mathbf{E}$ appears in the ith component, we mean that we only apply $\mathbf{E}$ move to the f_i parameterization and identity to all others. An example of such a move is shown on figure 22. For simplicity of the picture, we take three components.
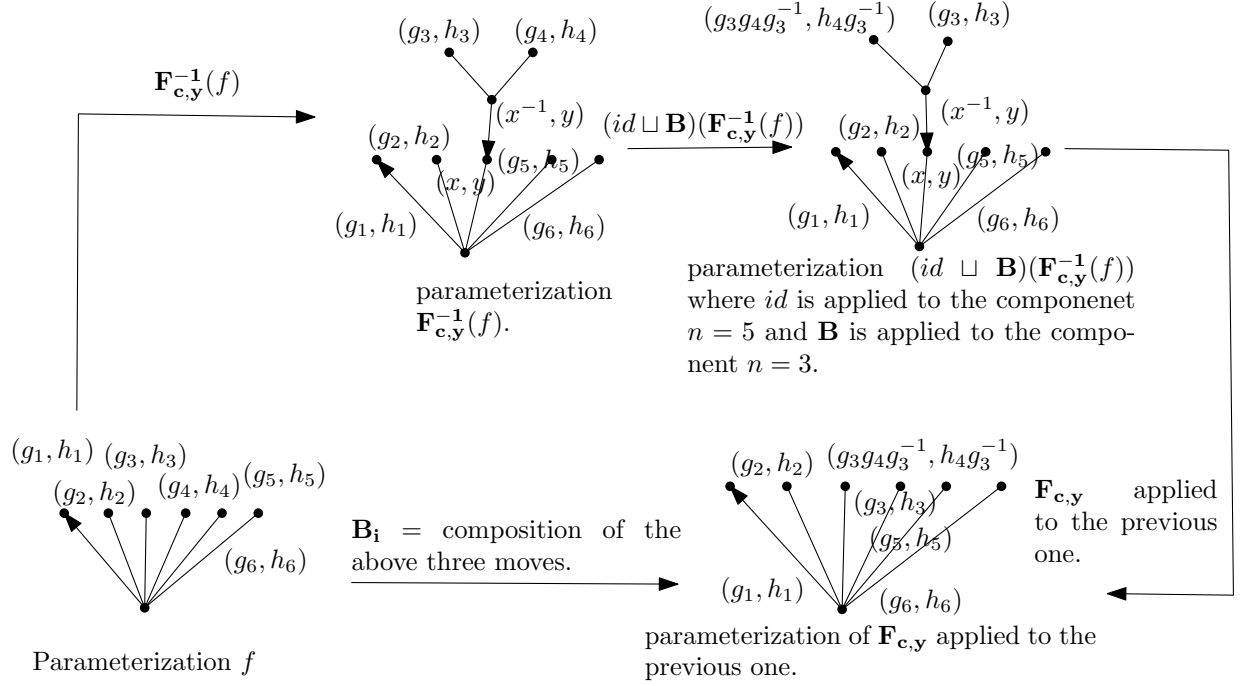
FIGURE 23. description of $\mathbf{B_i}$ move as a composition of previously
defined move

4.6. **$\mathbf{B_i}$ move.** The $\mathbf{B}$ move is very restricted; it is only defined for $n = 3$. So we
need a little flexibility. This $\mathbf{B_i}$ move will be a braiding of i and $i+1$ boundary
circles. This is in fact not a new move but a composition of previously defined
moves. This will be used many times from now on when we will describe the
other moves and relation. So it is important to do this right now. Let f be a
parameterization of a G cover, $\tilde{\Sigma}$, with a standard block say $S_n(g_1, ..., g_n; h_1, ..., h_n)$.
Then $\mathbf{B_i}$ is the composition of the moves described in the figure 23. Here we only
draw pictures for $n = 6$ for convenience but readers are clear of what should be
done for other n. If

$$\tilde{\Sigma} \xrightarrow{f} S_n(g_1, ..., g_n; h_1, ..., h_n)$$

Then

$$\tilde{\Sigma} \xrightarrow{\mathbf{B_i}(f)} S_n(g_1, .., g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, ..., g_n; h_1, .., h_{i-1}, h_{i+1} g_i^{-1}, h_i, .., h_n)$$

5. The Complex

5.1. **Definition of the complex $M(\tilde{\Sigma}, \Sigma)$.** We are given a fixed G-cover, $\Pi :$
$\tilde{\Sigma} \longrightarrow \Sigma$, where Σ will always denote an extended surface which is compact, ori-
entable, closed surface of genus zero unless otherwise specified. We will first define
the 1-skeleton of the complex, $M(\tilde{\Sigma}, \Sigma)$, that is the vertex and the edges.
{The set of vertex of $M(\tilde{\Sigma}, \Sigma)$} = {The set of parameterization of the G-cover
$\Pi : \tilde{\Sigma} \longrightarrow \Sigma$}

A vertex, α, is connected to vertex β, directed from α to β, if β can be obtained from α by applying one of the following moves:
$\mathbf{Z}, \mathbf{Z^{-1}}, \mathbf{B_i}, \mathbf{B_i^{-1}}, \mathbf{F}, \mathbf{F^{-1}}, \mathbf{P_x}, \mathbf{P_x^{-1}}, \mathbf{T_{z,y}^c}, (\mathbf{T_{z,y}^c})^{-1}$ in one of the components. Having defined the vertex and edges of the complex $M(\tilde{\Sigma}, \Sigma)$, now we need to define the relations or 2-cells of $M(\tilde{\Sigma}, \Sigma)$. We define the relations in the next section.

5.2. **The Statement Of the Main Theorem.**

Main Theorem. *The complex, $M(\tilde{\Sigma}, \Sigma)$, with the above defined edges and relations (which will be described in the next section), is connected and simply connected.*

6. Relations

Now it is time to define all the relations between these moves. These relation will be described in the following subsections:

6.1. **Obvious Relation.**

$$\mathbf{EE^{-1}} = \mathbf{E^{-1}E} = 1$$

Where $\mathbf{E}$ is one of the five moves, that is $\mathbf{E} \in \{\mathbf{Z}, \mathbf{B_i}, \mathbf{F}, \mathbf{P_x}, \mathbf{T_{z,y}^c}\}$

6.2. $\mathbf{P_x}$ **Relation.** $\mathbf{P_x}$ move commutes with all the other moves. More precisely for all $x, y \in G$ and all cuts, c, we have the following relations:

(1) $\mathbf{P_x Z} = \mathbf{ZP_x}$
(2) $\mathbf{P_x B_i} = \mathbf{B_i P_x}$
(3) $\mathbf{P_x F_{c,y}} = \mathbf{F_{c,yx^{-1}}}(\mathbf{P_x} \sqcup \mathbf{P_x})$. See figure 24.
(4) $\mathbf{P_x P_y} = \mathbf{P_{xy}}$

Remark 8. Note the change of indices for $\mathbf{F}$. Why we need to change the indices is clear from figure 24. Here we only label the edges corresponding to the cut c.

Remark 9. The last relation shows that whatever relation satisfied by the elements of G, the same relation hold for $\mathbf{P_x}$. So in particular this implies $(\mathbf{P_x})^{-1} = \mathbf{P_{x^{-1}}}$.

6.3. $\mathbf{P} - \mathbf{F}$ **relation.** Let f be a parameterization having two component glued to a cut c where one edge of the cut labeled by (x, y) and the other edge of the cut labeled by (z, w). For this glue to exist, we must have $yx^{-1}y^{-1} = [wz^{-1}w^{-1}]^{-1}$, see the subsection 3.5 on page 13. Let $t = w^{-1}y$. Then one can easily check that $txt^{-1} = z^{-1}$, $yt^{-1} = w$, $t^{-1}zt = x^{-1}$ and $wt = y$. Then $\mathbf{P} - \mathbf{F}$ relation is the following relation://

$$\mathbf{F_{c,w}}(\mathbf{P_t} \sqcup \mathbf{id}) = \mathbf{F_{c,y}}(\mathbf{id} \sqcup \mathbf{P_{t^{-1}}})$$

See the figure 25 for a visual description of this relation.

6.4. $\mathbf{Z}$ **Relation.** $\mathbf{Z}$ commutes with all the other moves. More precisely, let a parameterization, f, identifies a G cover, $\tilde{\Sigma}$, with the standard block $S_n(g_1, ..., g_n; h_1, ..., h_n)$. Then we have the following relation:

$$\mathbf{ZB_i} = \mathbf{B_{i+1}Z}$$

See figure 26 for a visual description of this relation.
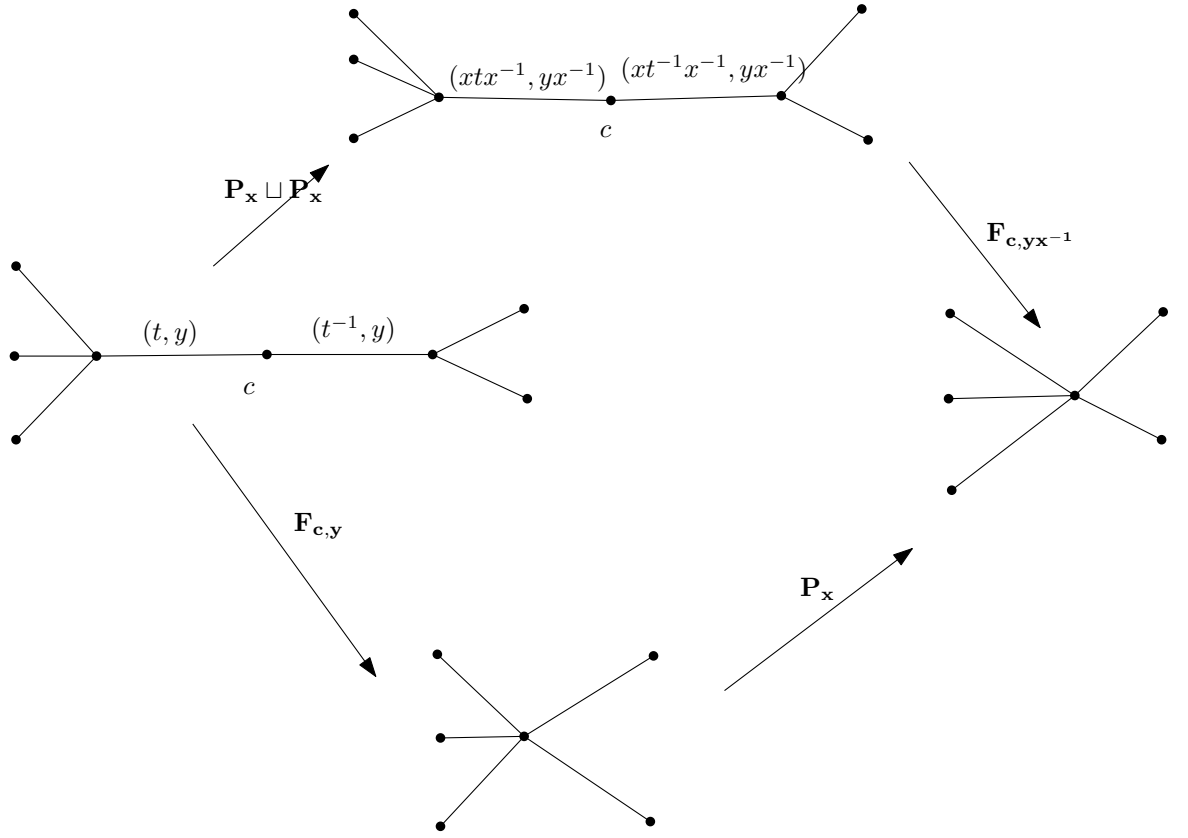
FIGURE 24. Visual description of the third $\mathbf{P_x}$ relation
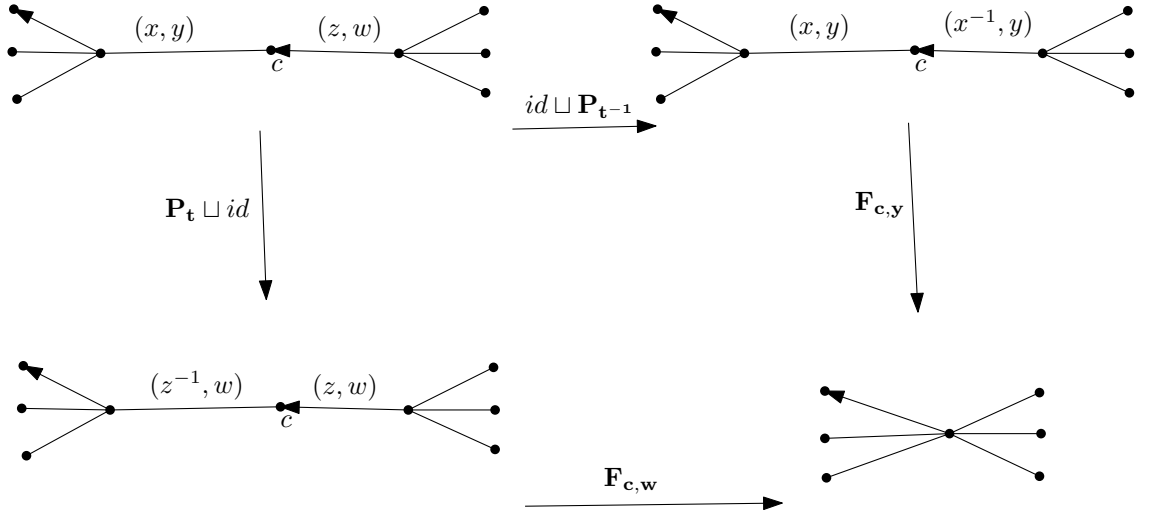
FIGURE 25. Visual description of $\mathbf{P} - \mathbf{F}$ relation
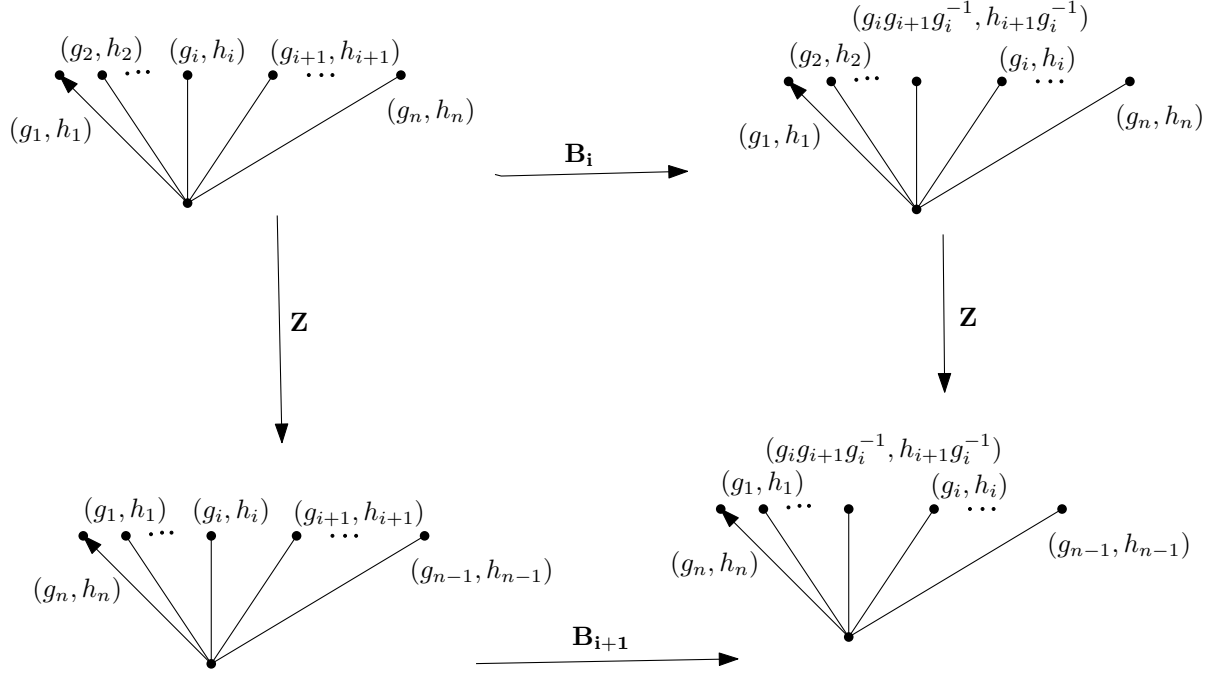
FIGURE 26. Notice the change of indices for $\mathbf{B}$.

Also let a parameterization, f, identifies a G cover, $\tilde{\Sigma}$, with gluing of two standard blocks along a cut c.

$$f : \tilde{\Sigma} \to S_n(g_1, ..., g_{n-1}, x; h_1, ..., h_{n-1}, y) \bigsqcup_{c,y} S_m(x^{-1}, g'_1, ..., g'_{m-1}; y, h'_1, ..., h'_{m-1})$$

Then we have the following relation:

$$(id \sqcup \mathbf{Z})\mathbf{T}^{\mathbf{c}}_{\mathbf{z},\mathbf{y}} = \mathbf{T}^{\mathbf{c}}_{\mathbf{z},\mathbf{y}}(id \sqcup \mathbf{Z})$$

See figure 27 for a visual description of this relation.

6.5. **B Relation.** From above relations, we already know that $\mathbf{B}$ move commutes with $\mathbf{Z}$ and $\mathbf{P_x}$ move. But we have more–$\mathbf{B}$ also commutes with the $\mathbf{T}$ move. More precisely, let a parameterization, f, identifies a G cover, $\tilde{\Sigma}$, with gluing of two standard blocks where one of the standard blocks has three boundary circles.

$$f : \tilde{\Sigma} \to S_n(g_1, ..., g_{n-1}, x; h_1, ..., h_{n-1}, y) \bigsqcup_{c,y} S_3(x^{-1}, g'_1, g'_2; y, h'_1, h'_2)$$

Here c denotes the cut where these two standard blocks are glued together. Then we have the following relation:

$$(id \sqcup \mathbf{B})\mathbf{T}^{\mathbf{c}}_{\mathbf{z},\mathbf{y}} = \mathbf{T}^{\mathbf{c}}_{\mathbf{z},\mathbf{y}}(id \sqcup \mathbf{B})$$

See figure 28 where for simplicity we take $n = 4$.

6.6. $\mathbf{T}^{\mathbf{c}}_{\mathbf{z},\mathbf{y}}$ **relation.** This is simply the relation $\mathbf{T}^{\mathbf{c}}_{\mathbf{z},\mathbf{y}} = \mathbf{F}^{-1}_{\mathbf{c},\mathbf{z}}\mathbf{F}_{\mathbf{c},\mathbf{y}}$ which is expected.
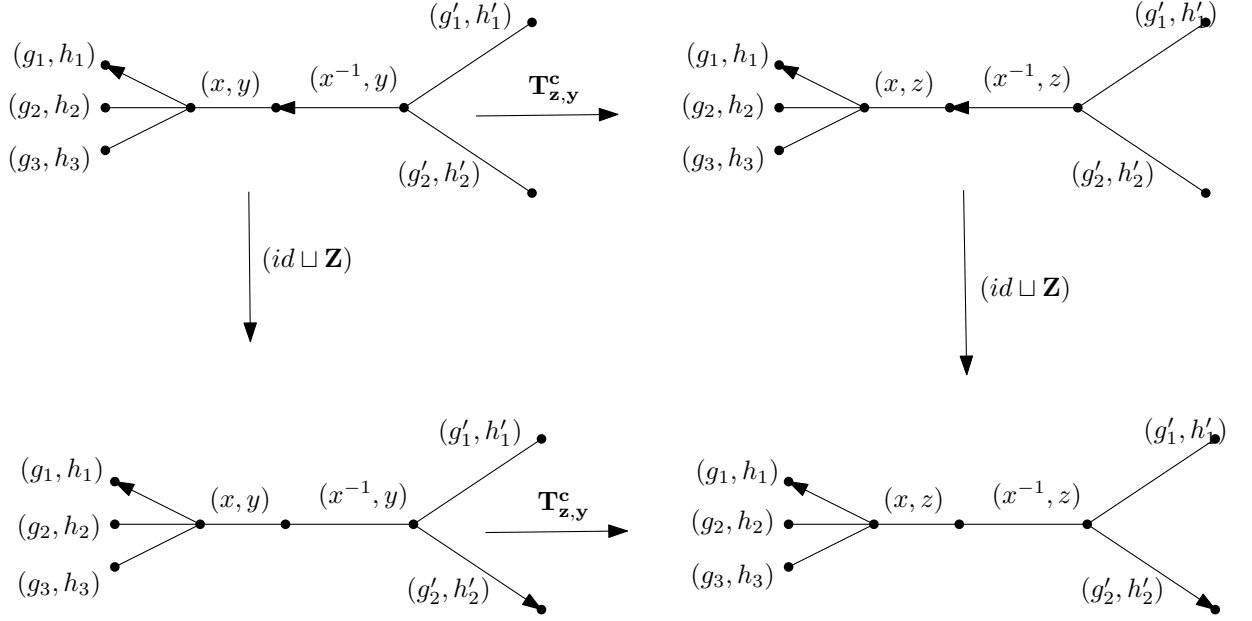
Here for simplicity of the picture, we take $n = 4$ and $m = 3$

FIGURE 27. The commutativity of $\mathbf{Z}$ and $\mathbf{T}$ moves
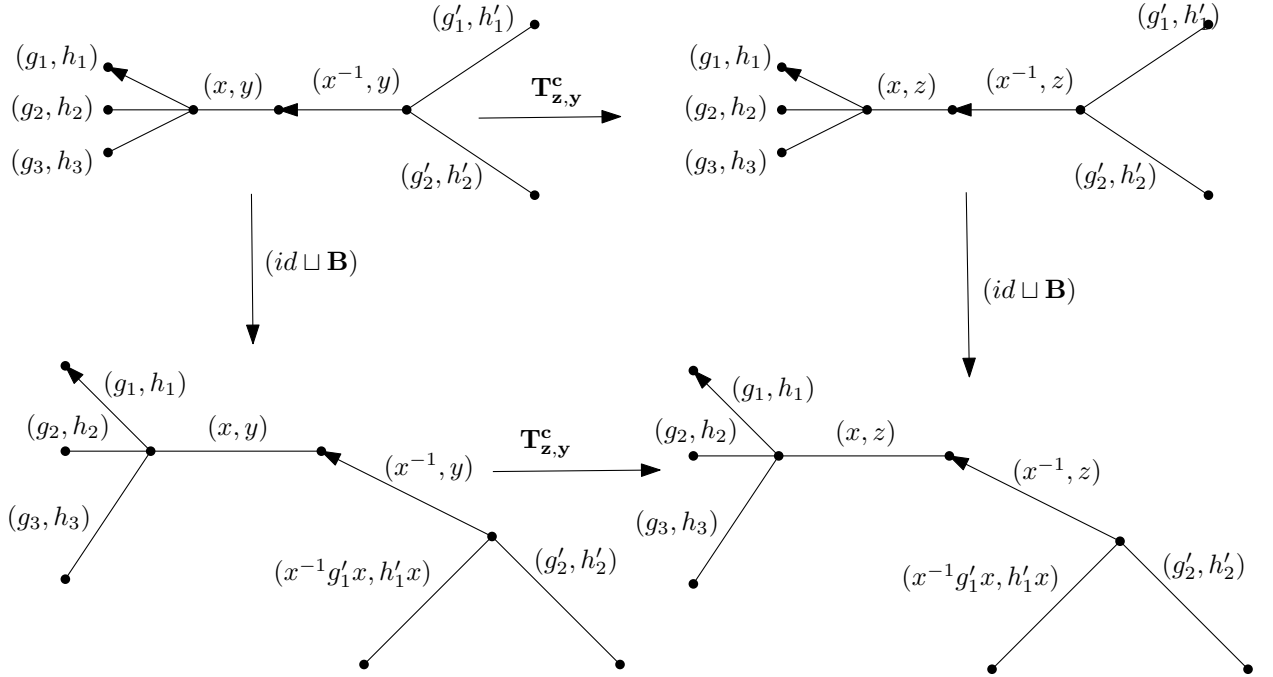
FIGURE 28. The commutativity of $\mathbf{B}$ and $\mathbf{T}$ moves
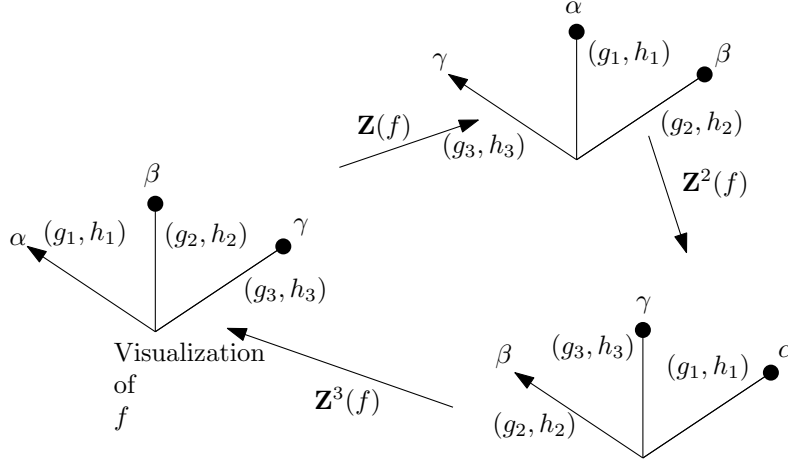
For simplicity of the picture, we take $n = 3$.

FIGURE 29. The 2-cell of $\mathbf{Z}$ relation, for $n = 3$

6.7. **Rotation axiom.** Let $f : \tilde{\Sigma} \longrightarrow S_n(g_1...g_n; h_1...h_n)$ be a parameterization of the G-cover, $\tilde{\Sigma}$. Then the rotation axiom says that if we apply $\mathbf{Z}$ move n times to the parameterization, f, we get back the same parameterization, f. That is $\mathbf{Z^n}f = f$ or $\mathbf{Z^n} = 1$. See the diagram on figure 29.

6.8. **Commutativity of Disjoint Union.** Let $\Sigma = \Sigma_1 \bigsqcup \Sigma_2$. Then by restriction, the G-cover $\tilde{\Sigma} \longrightarrow \Sigma$ can be broken down into two pieces, say,
$\tilde{\Sigma_1} \longrightarrow \Sigma_1$ and $\tilde{\Sigma_2} \longrightarrow \Sigma_2$.
Thus a parameterization of $\tilde{\Sigma} \longrightarrow \Sigma$ can be realized as disjoint union of two parameterization, one for $\tilde{\Sigma_1} \longrightarrow \Sigma_1$, and the other for $\tilde{\Sigma_2} \longrightarrow \Sigma_2$. We can reformulate this statement in the language of complex. In this case, it means that if A is a vertex of the complex $M(\tilde{\Sigma}, \Sigma)$, then A can be written as $A = (A_1, A_2)$ where A_1 is a vertex of $M(\tilde{\Sigma_1}, \Sigma_1)$ and A_2 is a vertex of $M(\tilde{\Sigma_2}, \Sigma_2)$. Let $\mathbf{E_i}$ be an edge of $M(\tilde{\Sigma_i}, \Sigma_i)$ directed from A_i to A'_i, here $i = 1, 2$. Then the commutativity of disjoint union is the following relation:

$$(id \bigsqcup \mathbf{E_2})(\mathbf{E_1} \bigsqcup id) = (\mathbf{E_1} \bigsqcup id)(id \bigsqcup \mathbf{E_2})$$
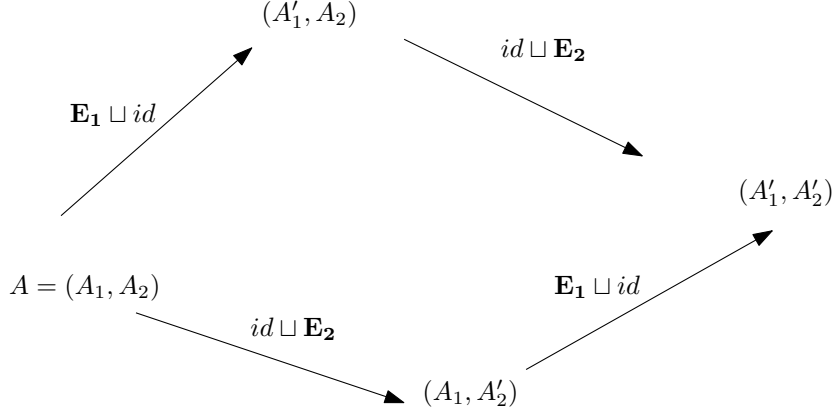
See the diagram on figure 30 for a visual presentation.

6.9. **Symmetry of F Move.** Let f be the following parameterization of the G-cover, $\tilde{\Sigma}$,

$$f : \tilde{\Sigma} \longrightarrow S_{k+1}(g_1, ...g_{k+1}; h_1, ...h_{k+1}) \bigsqcup_{c, h_{k+1}} S_{l+1}(g'_1, ...g'_{l+1}; h'_1, ...h'_{l+1})$$

Here the symbol,

$$\bigsqcup_{c, h_{k+1}}$$

$$(A_1', A_2)$$

$$\mathbf{E_1} \sqcup id \qquad\qquad\qquad\qquad id \sqcup \mathbf{E_2}$$

$$(A_1', A_2')$$

$$A = (A_1, A_2) \qquad\qquad\qquad\qquad \mathbf{E_1} \sqcup id$$

$$id \sqcup \mathbf{E_2}$$

$$(A_1, A_2')$$

Here $\mathbf{E_i}$ is an edge from A_i to A_i', and $i = 1, 2$

FIGURE 30. The 2-cell of commutativity of disjoint union

means that we glued along the cut c where the edges of the cut are labeled by (t, h_{k+1}) and (t^{-1}, h_{k+1}). This t is completely determined by all the other labels. More precisely we have the following:

The boundary component labeled by g_{k+1} of the standard block, $S_{k+1}(g_1, ...g_{k+1}; h_1, ...h_{k+1})$ is glued with the boundary component labeled by g_1' of the standard block $S_{l+1}(g_1', ...g_{l+1}'; h_1', ...h_{l+1}')$. of course we must have $h_{k+1} = h_1'$ and $g_{k+1} = (g_1')^{-1}$ for gluing to exist. From this parameterization, f, we can get an another parameterization, j, where

$$j : \tilde{\Sigma} \longrightarrow S_{l+1}(g_2', ..., g_{l+1}', g_1'; h_2', ..., h_{l+1}', h_1') \bigsqcup_{c, h_1'} S_{k+1}(g_{k+1}, g_1, ..., g_k; h_{k+1}, h_1, ..., h_k)$$

This is done by first interchange the component $S_{k+1}(g_1, ...g_{k+1}; h_1, ...h_{k+1})$ and $S_{l+1}(g_1', ..., g_{l+1}'; h_1', ..., h_{l+1}')$ and then we apply $\tilde{z}^{-1}$ to $S_{l+1}(g_1', ..., g_{l+1}'; h_1', ..., h_{l+1}')$ and $\tilde{z}$ to $S_{k+1}(g_1, ..., g_{k+1}; h_1, ..., h_{k+1})$ so that the boundary component labeled by g_1' becomes the last component of $S_{l+1}(g_1', ...g_{l+1}'; h_1', ...h_{l+1}')$ and the boundary component labeled by g_{k+1} becomes the first component of $S_{k+1}(g_1, ...g_{k+1}; h_1, ...h_{k+1})$. See Lemma 7 for the description of $\tilde{z}$

Now the symmetry of $\mathbf{F}$ move is the following relation:

$$\mathbf{Z^l F_{c, h_{k+1}}}(f) = \mathbf{F_{c, h_1'}}(j)$$

Or in short, we can just say

$$\mathbf{Z^l F_{c, h_{k+1}}} = \mathbf{F_{c, h_1'}}$$

Here c is the cut where we glue the G-cover. For the notation, $F_{c, h_{k+1}}$, see the remark on sec 6.2.3. In word, this relation just says that, we can first interchange the component, apply $\mathbf{Z}$ and $\mathbf{Z^{-1}}$ move to the components and then apply $\mathbf{F}$ move or we can just first apply the $\mathbf{F}$ move and then apply some appropriate power of $\mathbf{Z}$ move.

6.10. **Associativity of Cuts.** Let f be the following parameterization of the G-cover $\tilde{\Sigma}$
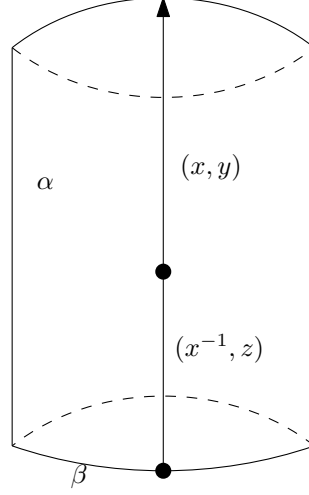
FIGURE 31. Standard marking on S_2

$$f : \tilde{\Sigma} \to S_{k+1}(g_1, ..g_{k+1}; h_1, ..h_{k+1}) \bigsqcup_{c_1, h_{k+1}} S_{l+1}(g'_1, ..g'_{l+1}; h'_1, ..h'_{l+1}) \bigsqcup_{c_2, h'_{l+1}} S_{n+1}(g''_1, ..g''_{n+1}; h''_1, ..h''_{n+1})$$

For the description of the notation,

$$\bigsqcup_{c_1, h_{k+1}} \text{ and } \bigsqcup_{c_2, h'_{l+1}}$$

see the relation "Symmetry of $\mathbf{F}$ move". Here c_1 denote the first cut and c_2 denote the second cut of the gluing. Of course, we assume that all the conditions are satisfied for gluing to exist. More specifically, we assume the following conditions:
$h_{k+1} = h'_1$, $h'_{l+1} = h''_1$, $g_{k+1}g'_1 = 1$, $g'_{l+1}g''_1 = 1$
Now the "associativity of cuts" is the following relation:

$$\mathbf{F}_{\mathbf{c_2}, \mathbf{h'_{l+1}}} \mathbf{F}_{\mathbf{c_1}, \mathbf{h_{k+1}}}(f) = \mathbf{F}_{\mathbf{c_1}, \mathbf{h_{k+1}}} \mathbf{F}_{\mathbf{c_2}, \mathbf{h'_{l+1}}}(f)$$

or in short, we can just say

$$\mathbf{F}_{\mathbf{c_2}, \mathbf{h'_{l+1}}} \mathbf{F}_{\mathbf{c_1}, \mathbf{h_{k+1}}} = \mathbf{F}_{\mathbf{c_1}, \mathbf{h_{k+1}}} \mathbf{F}_{\mathbf{c_2}, \mathbf{h'_{l+1}}}$$

In word, this just says that, given two distinct cuts, it does not matter in which order we apply the $\mathbf{F}$ move.

6.11. **Cylinder Axiom.** We consider the standard cylinder (S_2) with the standard marking. See the picture on figure 31. Let $f : \tilde{\Omega} \longrightarrow S_2(x, x^{-1}; y, z)$ be a parameterization of a G-cover, $\tilde{\Omega} \to \Omega$. Also let j be the following parameterization of G-cover, $\tilde{\Sigma} \to \Sigma$:

$$j : \tilde{\Sigma} \longrightarrow S_{k+1}(g_1, ..., g_k, x^{-1}; h_1, ..., h_k, y)$$

Note that, here we choose all the elements of G in such a way that we can glue the G-cover $\tilde{\Omega}$ and $\tilde{\Sigma}$. So after this gluing, we get a new parameterization of the
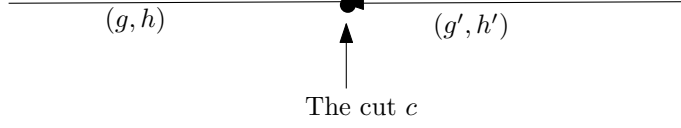
$$(g, h) \qquad\qquad (g', h')$$

The cut c

FIGURE 32.

G-cover

$$\tilde{\Sigma} \bigsqcup_{glued} \tilde{\Omega}$$

Let us denote this parameterization by $j \sqcup f$. That is

$$j \sqcup f : \tilde{\Sigma} \bigsqcup_{glued} \tilde{\Omega} \to S_{k+1}(g_1, ..., g_k, x^{-1}; h_1, ..., h_k, y) \bigsqcup_{x^{-1}, x} S_2(x, x^{-1}; y, z)$$

Finally let $\mathbf{E}$ be a move which can be applied to the parameterization j. In the language of complex this just means that $\mathbf{E}$ is one of the edges of the complex $M(\tilde{\Sigma}, \Sigma)$ starting from the vertex j. Here of course, $\mathbf{E}$ must be one of the following edges

$\mathbf{Z}, \mathbf{Z^{-1}}, \mathbf{B_i}, \mathbf{Bi^{-1}}, \mathbf{F}, \mathbf{F^{-1}}, \mathbf{T^c_{z,y}}, (\mathbf{T^c_{z,y}})^{-1}, \mathbf{P_x}$ or $\mathbf{P_x^{-1}}$. Then the cylinder axiom is the following relation or 2-cell:

$$\mathbf{EF}(j \sqcup f) = \mathbf{F}(\mathbf{E} \sqcup \mathbf{id})(j \sqcup f)$$

Or in short, we can just say

$$\mathbf{EF} = \mathbf{F}(\mathbf{E} \sqcup \mathbf{id})$$

6.12. **Braiding Axiom.** Before we define the braiding axiom, we need the following two definition:

6.12.1. *Generalized* $\mathbf{F}$ *move.* We know that before we can apply $\mathbf{F}$ move to a parameterization, there are some assumption we must satisfy. More precisely, if (g, h) is the label for the last boundary circle of one component, Σ_1, and (g', h') is the label for the first boundary circle of the other component, Σ_2, then to apply $\mathbf{F}$ move we must satisfy:

$gg' = 1$ and $h = h'$

See the picture on figure 32. If to start with the boundary circle associated with (g, h) is not the last one and the boundary circle associated with (g', h') is not the first one, we can apply appropriate $\mathbf{Z}$ move to Σ_1 and Σ_2 to get what we want. In other word, the generalized $\mathbf{F}$ move is the composition of the following moves:

$$\overrightarrow{\mathbf{F_{c,h}}} = \mathbf{F_{c,h}}(\mathbf{Z^a} \sqcup \mathbf{Z^b})$$

Note that we used an arrow above the generalized $\mathbf{F}$ move to distinguish it from the usual $\mathbf{F}$ move. Sometimes if there is no confusion, we will just use the notation $\mathbf{F_{c,h}}$ to denote a generalized $\mathbf{F}$ move. Here "a" and "b" is chosen appropriately so that (g, h)-boundary circle of Σ_1 becomes the last one and (g', h')-boundary circle of Σ_2 becomes the first one.
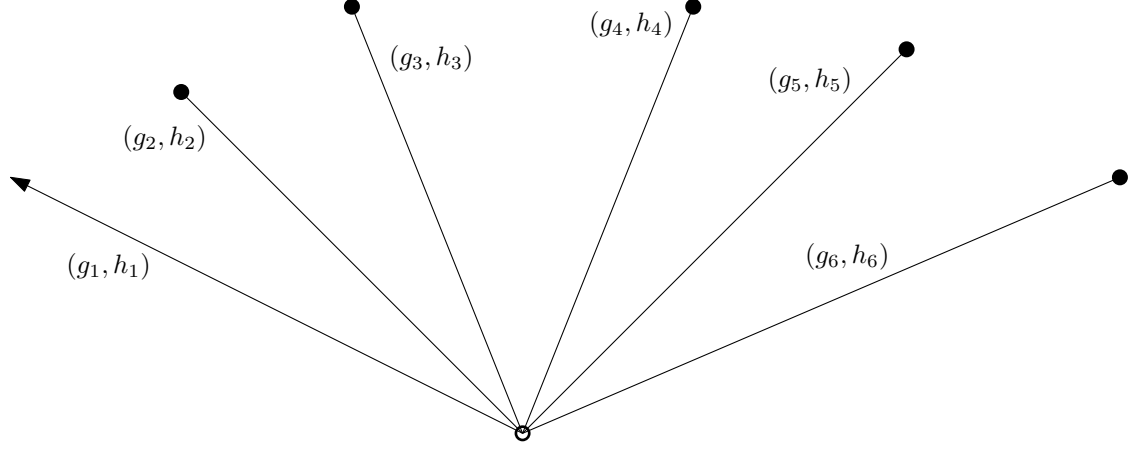
FIGURE 33. Visualization of f

Remark 10. In the case $a = b = 0$, the generalized $\overrightarrow{\mathbf{F_{c,h}}}$ move becomes the usual $\mathbf{F_{c,h}}$ move. So we can think the usual $\mathbf{F_{c,h}}$ move as a special case of generalized $\overrightarrow{\mathbf{F_{c,h}}}$ move.

6.12.2. *generalized Braiding move.* Let f be the following parameterization of the G-cover, $\tilde{\Sigma}$,

$$f : \tilde{\Sigma} \to S_{(k+l+n+m)}(g_1, ..., g_{k+l+n+m}; h_1, ...h_{k+l+n+m})$$

We also let

$I_1 = \{$the set of boundary circle associated with $(g_1, ...g_k)\}$

$I_2 = \{$the set of boundary circle associated with $(g_{k+1}, ...g_{k+l})\}$

$I_3 = \{$the set of boundary circle associated with $(g_{k+l+1}, ...g_{k+l+n})\}$

$I_4 = \{$the set of boundary circle associated with $(g_{k+l+n+1}, ...g_{k+l+n+m})\}$

We want to define the generalized $\mathbf{B}$ move denoted by $\mathbf{B_{I_2, I_3}}$. The best way to explain this is through an example. For simplicity, we take $|I_1| = 1, |I_2| = 2, |I_3| = 2, |I_4| = 1$, but readers easily see that it works in general. Now without further delay, we define the generalized $\mathbf{B_{I_2, I_3}}$ as the composition of the following moves: First of all we are given:

$f : \tilde{\Sigma} \to S_6(\underbrace{g_1}_{I_1}, \underbrace{g_2, g_3}_{I_2}, \underbrace{g_4, g_5}_{I3}, \underbrace{g_6}_{I_4}; h_1, h_2, h_3, h_4, h_5, h_6)$. See figure 33 for the visualization of f. In what follows, all the $\mathbf{F}$ and the $\mathbf{F^{-1}}$ move will be generalized $\overrightarrow{\mathbf{F}}$ and $(\overrightarrow{\mathbf{F_{c,h}}})^{-1}$ move, but we do not put arrow sign on top to make the picture simple. we apply $\mathbf{F_{c_3, w}^{-1} F_{c_2, v}^{-1} F_{c_1, y}^{-1}}$ to the parameterization f to get the following parameterization of $\tilde{\Sigma}$

$$\mathbf{F_{c_3, w}^{-1} F_{c_2, v}^{-1} F_{c_1, y}^{-1}}(f) : \tilde{\Sigma} \longrightarrow S_3(g_1, x, g_6; h_1, y, h_6) \bigsqcup_{(c_1, y)} S_3(x^{-1}, z, u; y, w, v)$$

$$\bigsqcup_{(c_2, v)} S_3(u^{-1}, g_4, g_5; v, h_4, h_5) \bigsqcup_{(c_3, w)} S_3(z^{-1}, g_2, g_3; w, h_2, h_3)$$
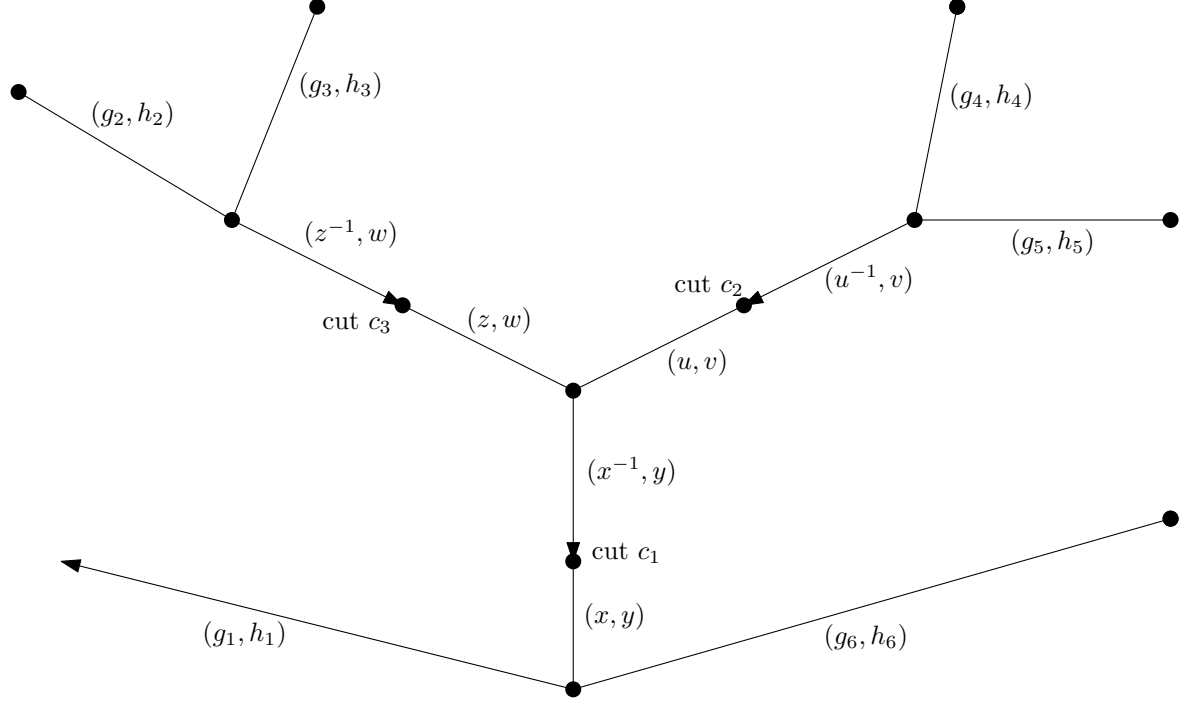
FIGURE 34. Visualization of $q = \mathbf{F}_{\mathbf{c_3},\mathbf{w}}^{-1}\mathbf{F}_{\mathbf{c_2},\mathbf{v}}^{-1}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}^{-1}(f)$

For simplicity of writing things down let us denote
$q = \mathbf{F}_{\mathbf{c_3},\mathbf{w}}^{-1}\mathbf{F}_{\mathbf{c_2},\mathbf{v}}^{-1}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}^{-1}(f)$. See figure 34. Now to the parameterization q, we apply
the move $id \sqcup$ (the usual $\mathbf{B}$ move on the edge labeled by (u,v) and $(z,w))\sqcup\mathbf{P_z} \sqcup id$
Recall that $\mathbf{P_z}$ means the $\mathbf{P}$ move for the element $z \in G$. See section 6.3 for detail.
let us denote by r the parameterization $\{id\sqcup$(the usual $\mathbf{B}$ move on the edge labeled
by (u,v) and $(z,w))\sqcup\mathbf{P_z} \sqcup id\}(q)$
Here

$$r : \tilde{\Sigma} \longrightarrow S_3(g_1, x, g_6; h_1, y, h_6) \bigsqcup_{(c_1,y)} S_3(x^{-1}, zuz^{-1}, z; y, vz^{-1}, w)$$

$$\bigsqcup_{c_2,vz^{-1}} S_3(zu^{-1}z^{-1}, zg_4z^{-1}, zg_5z^{-1}; vz^{-1}, h_4z^{-1}, h_5z^{-1}) \bigsqcup_{c_3,w} S_3(z^{-1}, g_2, g_3; w, h_2, h_3)$$

See the diagram on figure 35. Finally, to get rid off all the cuts, c_1, c_2, c_3, that have
been artificially created, we apply the move, $\mathbf{F}_{\mathbf{c_3},\mathbf{vz}^{-1}}\mathbf{F}_{\mathbf{c_2},\mathbf{w}}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}$, to r. Thus

$$\mathbf{F}_{\mathbf{c_3},\mathbf{vz}^{-1}}\mathbf{F}_{\mathbf{c_2},\mathbf{w}}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}(r) : \tilde{\Sigma} \longrightarrow S_6(g_1, zg_4z^{-1}, zg_5z^{-1}, g_2, g_3, g_6; h_1, h_4z^{-1}, h_5z^{-1}, h_2, h_3, h_6)$$

See the figure 36. This is the end of the generalized $\mathbf{B_{I_2,I_3}}$ move. If we put every-
thing together, we have the following composition for $\mathbf{B_{I_2,I_3}}$ move:
$\mathbf{B_{I_2,I_3}}(f) = \mathbf{F}_{\mathbf{c_3},\mathbf{vz}^{-1}}\mathbf{F}_{\mathbf{c_2},\mathbf{w}}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}\{id \sqcup$ (the usual $\mathbf{B}$ move on the edge labeled by
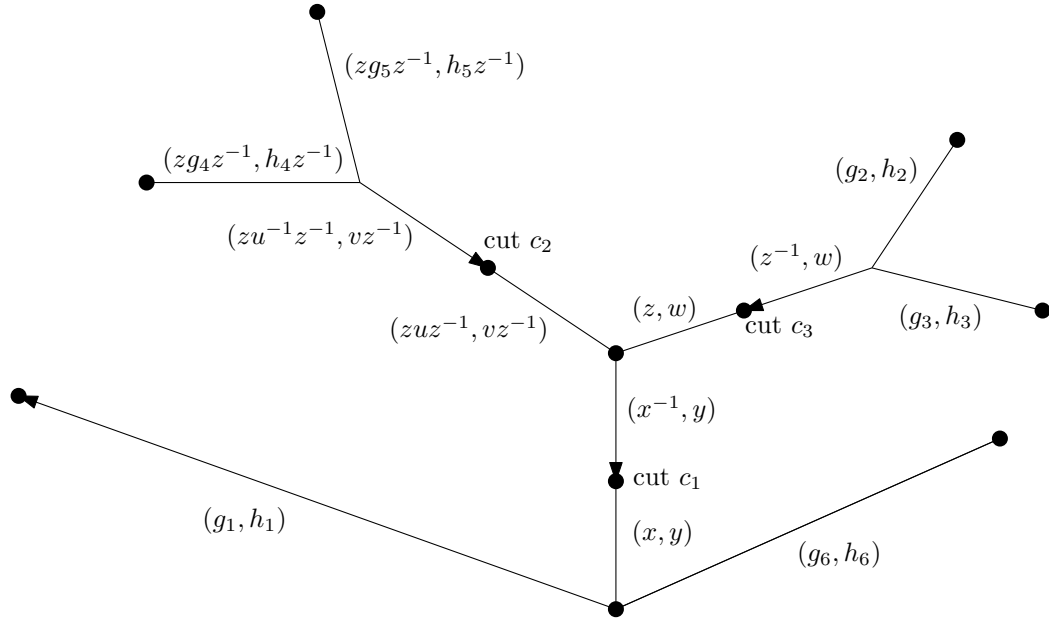(u,v) and $(z,w))\sqcup\mathbf{P_z} \sqcup id\}\mathbf{F}_{\mathbf{c_3},\mathbf{w}}^{-1}\mathbf{F}_{\mathbf{c_2},\mathbf{v}}^{-1}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}^{-1}(f)$

FIGURE 35. Visualization of r
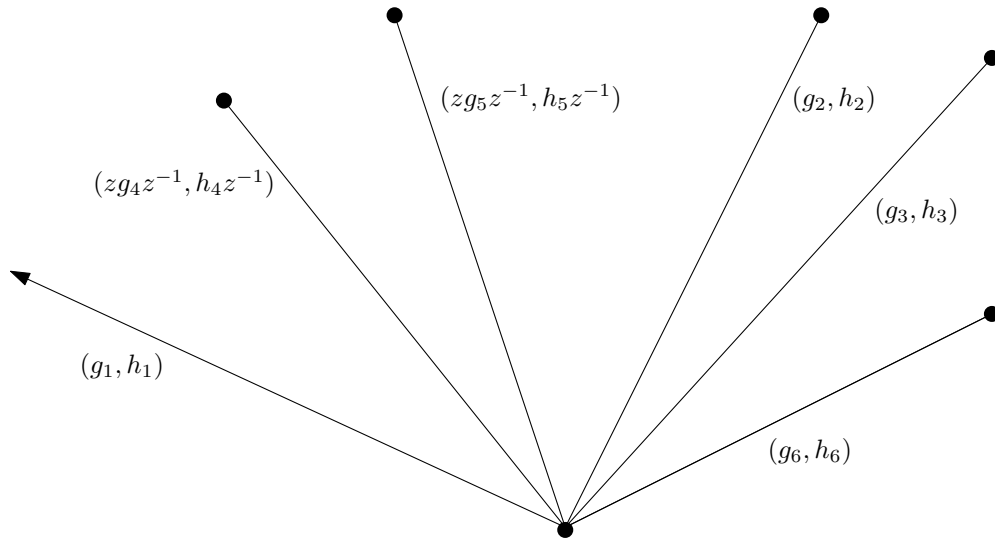
FIGURE 36. Visualization of $\mathbf{F}_{\mathbf{c_3},\mathbf{vz^{-1}}}\mathbf{F}_{\mathbf{c_2},\mathbf{w}}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}(r)$

Remark 11. Note that we did not use any parenthesis to denote the move $\mathbf{F}_{\mathbf{c_3},\mathbf{vz^{-1}}}\mathbf{F}_{\mathbf{c_2},\mathbf{w}}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}$ or $\mathbf{F}_{\mathbf{c_3},\mathbf{w}}^{-1}\mathbf{F}_{\mathbf{c_2},\mathbf{v}}^{-1}\mathbf{F}_{\mathbf{c_1},\mathbf{y}}^{-1}$ since by the "associativity of cuts", it does not matter on which order we add or remove cuts.

Remark 12. We might wonder, what is z that appear on the generalized $\mathbf{B}$ move above? We can in fact calculate the value of z quite easily.

$xg_6g_1 = 1 \Rightarrow x = g_1^{-1}g_6^{-1}$
$ux^{-1}g_2g_3 = 1 \Rightarrow u = g_3^{-1}g_2^{-1}g_1^{-1}g_6^{-1}$
$zux^{-1} = 1 \Rightarrow z = xu^{-1} = g_1^{-1}g_6^{-1}g_6g_1g_2g_3 = g_2g_3$
So $z = g_2g_3$ In general $z = $ multiplication of all g's in $\mathbf{I_2}$.

Remark 13. The generalized braiding move agree with the usual braiding move in the case when $|\mathbf{I_2}| = |\mathbf{I_3}| = 1$.

6.12.3. *Braiding Axiom.* Now it is time to describe the braiding axiom. We start with the following parameterization of our G-cover:
$f : \tilde{\Sigma} \longrightarrow S_4(g_1, g_2, g_3, g_4; h_1, h_2, h_3, h_4)$ where we name the boundary circle $\delta, \alpha, \beta, \gamma$, in the increasing order; that is δ is the boundary circle associated with (g_1, h_1) and γ is the boundary circle associated with (g_4, h_4).
Then the Braiding axiom is the following two relation:

$$\mathbf{B}_{\alpha,\gamma}\mathbf{B}_{\alpha,\beta}(f) = \mathbf{B}_{\{\alpha\},\{\beta,\gamma\}}(f)$$

and

$$\mathbf{B}_{\alpha,\gamma}\mathbf{B}_{\beta,\gamma}(f) = \mathbf{B}_{\{\alpha\beta\},\{\gamma\}}(f)$$

Here $\mathbf{B}_{\alpha,\gamma}$ etc denote the usual $\mathbf{B}$ move and $\mathbf{B}_{\{\alpha\},\{\beta,\gamma\}}$ etc denote the generalized $\mathbf{B}$ move.
We will describe step by step move for the first relation of Braiding Axiom since the description of the other relation is similar.

$$\mathbf{B}_{\alpha,\beta}(f) : \tilde{\Sigma} \longrightarrow S_6(g_1, g_2g_3g_2^{-1}, g_2, g_4; h_1, h_3g_2^{-1}, h_2, h_4)$$
$$\mathbf{B}_{\alpha,\gamma}\mathbf{B}_{\alpha,\beta}(f) : \tilde{\Sigma} \longrightarrow S_6(g_1, g_2g_3g_2^{-1}, g_2g_4g_2^{-1}, g_2; h_1, h_3g_2^{-1}, h_4g_2^{-1}, h_2)$$

Look at the definition of $\mathbf{B}$ move above. On the other hand
$$\mathbf{B}_{\{\alpha\},\{\beta,\gamma\}}(f) : \tilde{\Sigma} \longrightarrow S_6(g_1, g_2g_3g_2^{-1}, g_2g_4g_2^{-1}, g_2; h_1, h_3g_2^{-1}, h_4g_2^{-1}, h_2)$$

See the diagram on figure 37 for a visual presentation.

6.13. **Dehn Twist Axiom.** Say, we are given a Standard Cylinder (S_2) with the standard graph (see the picture below) where we denote by α, the first boundary component and by β, the second boundary component. Also let f be a parameterization of our G-cover, $\tilde{\Sigma}$, given by
$f : \tilde{\Sigma} \longrightarrow S_2(g, g^{-1}; h_1, h_2)$
Then the dehn twist axiom is the following relation:

$$\mathbf{Z}\mathbf{B}_{\alpha,\beta}(f) = \mathbf{P_g}\mathbf{B}_{\beta,\alpha}\mathbf{Z}(f)$$

Here $\mathbf{B}_{\alpha,\beta}$ denote the usual $\mathbf{B}$ move and $\mathbf{P_g}$ denote the $\mathbf{P}$ move for the element $g \in G$. For a detail description of these moves, see section 6. For a better understanding, we breakdown this relation piece by piece:
$$\mathbf{B}_{\alpha,\beta}(f) : \tilde{\Sigma} \longrightarrow S_2(gg^{-1}g^{-1}, g; h_2g^{-1}, h_1) = S_2(g^{-1}, g; h_2g^{-1}, h_1)$$
$$\mathbf{Z}\mathbf{B}_{\alpha,\beta}(f) : \tilde{\Sigma} \longrightarrow S_2(g, g^{-1}; h_1, h_2g^{-1})$$
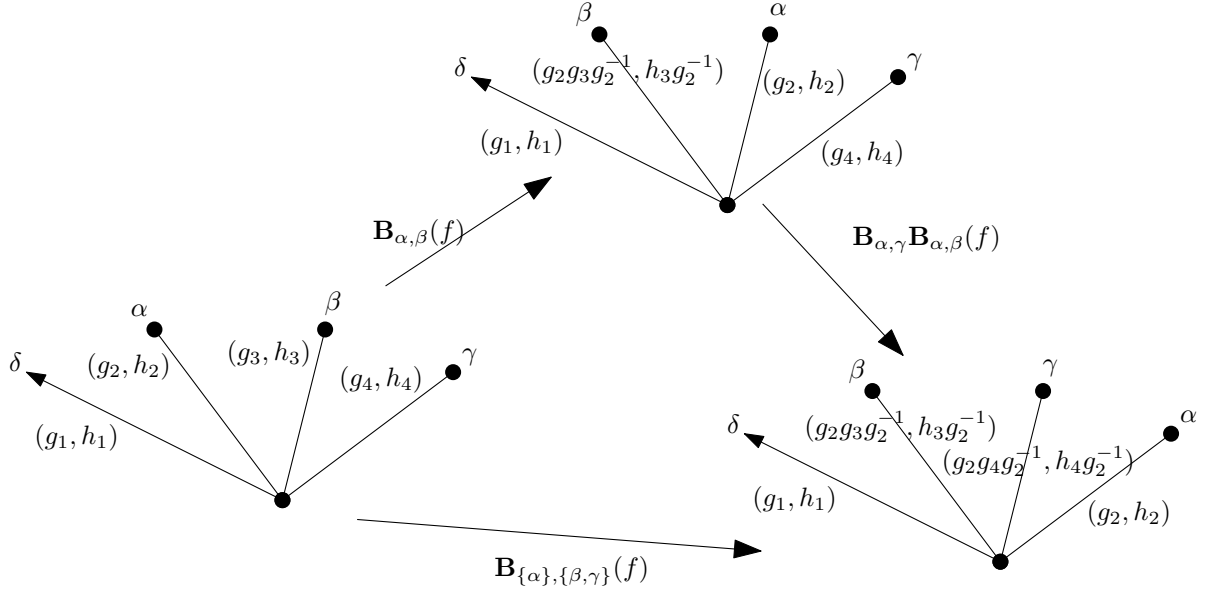
FIGURE 37. 2-cell or relation of braiding axiom

On the other hand

$$\mathbf{Z}(f) : \tilde{\Sigma} \longrightarrow S_2(g^{-1}, g; h_2, h_1)$$

$$\mathbf{B}_{\beta,\alpha}\mathbf{Z}(f) : \tilde{\Sigma} \longrightarrow S_2(g^{-1}gg, g^{-1}; h_1g, h_2) = S_2(g, g^{-1}; h_1g, h_2)$$

$$\mathbf{P_g}\mathbf{B}_{\beta,\alpha}\mathbf{Z}(f) : \tilde{\Sigma} \longrightarrow S_2(ggg^{-1}, gg^{-1}g^{-1}; h_1gg^{-1}, h_2g^{-1}) = S_2(g, g^{-1}; h_1, h_2g^{-1})$$

See the diagram on figure 38 for a visual description of the Dehn Twist Axiom.

Remark 14. Note that, here this dehn twist axiom differs from the dehn twist axiom of the paper [BK] (the case when $|G| = 1$), because of this extra $\mathbf{P_g}$ move appearing in the relation.

We conclude this section with the following lemma

Lemma 9. *All the relations describe above make sense. In other word, all the relation describe closed loop in our complex, $M(\tilde{\Sigma}, \Sigma)$.*

Proof. This is basically done when we explained the relation above. We can review each relation one by one and infact see that they are closed loop in our complex. The detail are left to the reader.   □

7. PROVING THE MAIN RESULT

7.1. **A general Theorem about Complexes.**

Theorem 2. *We will use this result to prove our main theorem. Let A and B be two 2-dim complex. Let $\Pi : B^{[1]} \longrightarrow A^{[1]}$ be a map of their 1-skeleton(vertices and edges), which is surjective both on vertices and edges. Also suppose the following condition is satisfied:*
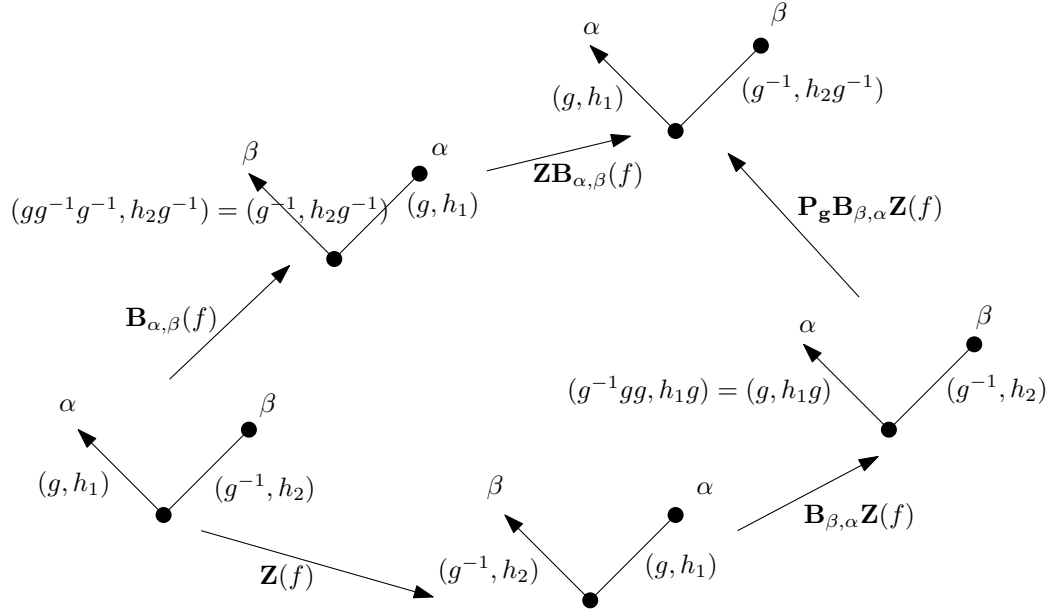
FIGURE 38. 2-cell or relation of Dehn twist axiom

(1) *A is connected and simply-connected*
(2) *For every vertex $a \in A$, $\Pi^{-1}(a)$ is connected and simply-connected. That is every loop, completely lie in $\Pi^{-1}(a)$, is contractable in B.*
(3) *Let*

$$b_1' \xrightarrow{f'} b_2'$$

and

$$b_1'' \xrightarrow{f''} b_2''$$

be two lifting of

$$a_1 \xrightarrow{f} a_2$$

Then there is a path e_1, starting from b_1'' and end with b_1', and completely lie in $\Pi^{-1}(a_1)$ and a path e_2, starting from b_2'' and end with b_2', and completely lie in $\Pi^{-1}(a_2)$, so that the following relation hold:

$$e_2 f'' = f' e_1$$

In other word, the following diagram commute:

$$
\begin{array}{ccc}
b_1'' & \xrightarrow{f''} & b_2'' \\
e_1 \downarrow & & \downarrow e_2 \\
b_1' & \xrightarrow{f'} & b_2'
\end{array}
$$

(4) *Every loop in A can be lifted to a contractible loop in B.*

Then the complex B is connected and simply-connected.

Proof. Not hard and we will leave the proof to the reader. □

7.2. **Proof Of the Main Theorem.** Recall that the main theorem says that the complex $M(\tilde{\Sigma}, \Sigma)$ is connected and simply-connected. To use the previous theorem, we let
$A = M(\Sigma):=$ (as defined in the paper [BK].)

- Vertex of $M(\Sigma)$ = Marking with cuts
- Edges of $M(\Sigma)$ = Z, B, F move described in the paper [BK]
- Relation = Described in the paper [BK]

We also let $B = M(\tilde{\Sigma}, \Sigma) :=$ see section 7 for the definition of this complex.

7.2.1. *Description of* $\Pi : M(\tilde{\Sigma}, \Sigma)^{[1]} \to M(\Sigma)^{[1]}$. We first describe the map of 1-skeleton. If v is a vertex in $M(\tilde{\Sigma}, \Sigma)$, then v is given by marked graph, with (g_i, h_i) in each cut. Π just take this vertes v to the marked graph, forgetting about (g_i, h_i). This marked graph without (g_i, h_i) is a vertex in $M(\Sigma)$
If $\mathbf{E}$ is an edge in $M(\tilde{\Sigma}, \Sigma)$, where $\mathbf{E}$ is one of $\mathbf{Z}, \mathbf{B_i}$ or $\mathbf{F}$ move, then $\Pi(\mathbf{E})$ is the corresponding Z, B_i and F move in $M(\Sigma)$. Since a $\mathbf{P_g}, g \in G$ and $\mathbf{T^c_{z,y}}$ move does not change the underling marked graph, $\Pi(\mathbf{P_g}) = \Pi(\mathbf{T^c_{z,y}}) =$ empty edge. This conclude the description of Π. Note that Π is obviously surjective both on the vertex and edges.

Lemma 10. *The above map of complex,* $\Pi : M(\tilde{\Sigma}, \Sigma) \to M(\Sigma)$ *make sense. In other word,* Π *does define a map of complex.*

Proof. obvious. □

7.2.2. *Description of a typical fiber.* Let $a \in$ vertex $(M(\Sigma))$. We want to describe $\Pi^{-1}(a)$. Hence the following lemma:

Lemma 11. *Let b and $b^{'} \in \Pi^{-1}(a)$. Then b and $b^{'}$ is connected by a sequence of moves of the form $\mathbf{F^{-1}_{c_i,z}}\mathbf{F_{c_i,y}} := \mathbf{T^{c_i}_{z,y}}$ and $\mathbf{P_x}$. See section 6 for the description of these moves and the notation. Here c_i is the i-th cut and $z, y, x \in G$. Also conversely, only move that takes a vertex of a fiber to the same fiber, is the move $\mathbf{T^{c_i}_{z,y}}$ and $\mathbf{P_x}$.*

Proof. Let the vertex $a \in M(\Sigma)$ be parameterized by k cuts. That is

$$\Sigma = \Sigma_1 \underset{\text{glued}}{\bigsqcup} \Sigma_2 \underset{\text{glued}}{\bigsqcup} \ldots\ldots \underset{\text{glued}}{\bigsqcup} \Sigma_k$$

Then the corresponding G-cover can also be break down as a gluing of k G-cover. That is

$$\tilde{\Sigma} = \tilde{\Sigma}_1 \underset{\text{glued}}{\bigsqcup} \tilde{\Sigma}_2 \underset{\text{glued}}{\bigsqcup} \ldots\ldots \underset{\text{glued}}{\bigsqcup} \tilde{\Sigma}_k$$

Then the parameterization, b and $b^{'}$ can be realized as a gluing of k parameterization; each one coming from the parameterization of $\tilde{\Sigma}_i \longrightarrow \Sigma_i$ which we will denote

by b_i and b_i'. So in short hand notation we can write $b = b_1 \sqcup b_2 \sqcup ... \sqcup b_k$ and $b' = b_1' \sqcup b_2 \sqcup ... \sqcup b_k'$. Here $i = 1...k$. Let

$$b : \tilde{\Sigma} \longrightarrow S_{n_1}(g_1^1, ...g_{n_1}^1; h_1^1, ...h_{n_1}^1) \underset{c_1, h_{n_1}^1}{\bigsqcup} ..... \underset{c_k, h_{n_{k-1}}^{k-1}}{\bigsqcup} S_{n_k}(g_1^k, ...g_{n_k}^k; h_1^k, ...h_{n_k}^k)$$

and let

$$b' : \tilde{\Sigma} \longrightarrow S_{n_1}(p_1^1, ...p_{n_1}^1; q_1^1, ...q_{n_1}^1) \underset{c_1, q_{n_1}^1}{\bigsqcup} ..... \underset{c_k, q_{n_{k-1}}^{k-1}}{\bigsqcup} S_{n_k}(p_1^k, ...p_{n_k}^k; q_1^k, ...q_{n_k}^k)$$

consider the parameterization

$$b_1 : \tilde{\Sigma}_1 \longrightarrow S_{n_1}(g_1^1, ...g_{n_1}^1; h_1^1, ...h_{n_1}^1)$$

and

$$b_1' : \tilde{\Sigma}_1 \longrightarrow S_{n_1}(p_1^1, ...p_{n_1}^1; q_1^1, ...q_{n_1}^1)$$

Then

$$b_1'b_1^{-1} : S_{n_1}(g_1^1, ...g_{n_1}^1; h_1^1, ...h_{n_1}^1) \longrightarrow S_{n_1}(p_1^1, ...p_{n_1}^1; q_1^1, ...q_{n_1}^1)$$

is an isomorphism of G-cover. So from lemma 3, we know that there exist an $x_1 \in G$ so that $S_{n_1}(x_1 g_1^1 x_1^{-1}, ...x_1 g_{n_1}^1 x_1^{-1}; h_1^1 x_1^{-1}, ...h_{n_1}^1 x_1^{-1}) = S_{n_1}(p_1^1, ...p_{n_1}^1; q_1^1, ...q_{n_1}^1)$
Similarly we can find $x_2, x_3, ...x_k$ so that $x_i g_j^i x_i^{-1} = p_j^i$ and $h_j^i x_i^{-1} = q_j^i$ for $i = 2...k$.
Now let us apply the $\mathbf{P_{x_1}} \sqcup .... \sqcup \mathbf{P_{x_k}}$ move to b, Then

$$\mathbf{P_{x_1}} \sqcup .... \sqcup \mathbf{P_{x_k}}(b) : \tilde{\Sigma} \longrightarrow S_{n_1}(x_1 g_1^1 x_1^{-1}, ...x_1 g_{n_1}^1 x_1^{-1}; h_1^1 x_1^{-1}, ...h_{n_1}^1 x_1^{-1}) \underset{c_1, h_{n_1}^1 x_1^{-1}}{\bigsqcup} ..... \underset{c_k, h_{n_{k-1}}^{k-1} x_{k-1}^{-1}}{\bigsqcup}$$

$$S_{n_k}(x_k g_1^k x_k^{-1}, ...x_k g_{n_k}^k x_k^{-1}; h_1^k x_k^{-1}, ...h_{n_k}^k x_k^{-1})$$

By the choice of $x_1, x_2, ..., x_k$, we must have, $x_i g_i^j x_i^{-1} = p_i^j$ and $h_j^i x_i^{-1} = q_j^i$ as $j = 1...k$ and $i = 1...n_j$.
In other word, $\mathbf{P_{x_1}} \sqcup .... \sqcup \mathbf{P_{x_k}}(b) = b'$.

Now conversely, let $b \in \Pi^{-1}(a)$ and $\mathbf{E}$ is a move in $M(\tilde{\Sigma}, \Sigma)$, so that $\mathbf{E}(b) \in \Pi^{-1}(a)$. At first site, of course, all possibilities for $\mathbf{E}$ are $\mathbf{Z}, \mathbf{Z^{-1}B}, \mathbf{B^{-1}F}, \mathbf{F^{-1}}$ $\mathbf{P}, \mathbf{P^{-1}}, \mathbf{T}$ and $\mathbf{T^{-1}}$. But the move $\mathbf{Z}, \mathbf{Z^{-1}B}, \mathbf{B^{-1}F}, \mathbf{F^{-1}}$ will take the vertex, b, outside the fiber $\Pi^{-1}(a)$. So the only possible value for $\mathbf{E}$ is $\mathbf{P_x}$ and $\mathbf{T^c_{z,y}}$ move and their inverses. But $\mathbf{P_x^{-1}} = \mathbf{P_{x^{-1}}}$ and $(\mathbf{T^c_{z,y}})^{-1} = (\mathbf{F^{-1}_{c,z}F_{c,y}})^{-1} = \mathbf{F^{-1}_{c,y}F_{c,z}} = \mathbf{T^c_{y,z}}$. So the inverse of $\mathbf{P}, \mathbf{T}$ move are another $\mathbf{P}, \mathbf{T}$ move. $\square$

Remark: This lemma shows in particular that the fiber $\Pi^{-1}(a)$ is connected.

7.2.3. *The fiber $\Pi^{-1}(a)$ is simply-connected.*

Lemma 12. *The fiber $\Pi^{-1}(a)$ is simply-connected. That is given any loop, where each vertex of the loop belong to $\Pi^{-1}(a)$, is contractable using the 2-cell or relations of $M(\tilde{\Sigma}, \Sigma)$.*

Proof. We divide the proof in three cases.
case 1: no cuts and no boundary circle
Here the base surface, Σ, is isomorphic to S_0, the sphere, and the G-cover of Σ is trivial that is $\Sigma \times G$. This case must be treated separately. Although there is only one G-cover of Σ up to isomorphism, there are a total of $|G|$ many parameterization of this G-cover. To see this, recall that a parameterization, f, of our G-cover $\Sigma \times G$

is just an isomorphism from the G-cover $\Sigma \times G$ to the G-cover $S_0 \times G$. of course this f must maps a component of $\Sigma \times G$ isomorphically to a component of $S_0 \times G$. Let say that

$$f(\Sigma \times 1, \text{ where } 1 \in G) = S_0 \times h, \text{ where } h \in G$$

Then this information will determine f completely since f must preserve the action of G on the fiber. More precisely we have:

$$f(\Sigma \times x, \text{ where } x \in G) = S_0 \times xh$$

So a parameterization is completely determine by an element of $h \in G$. So in this way, we can identify the set of parameterization of our G-cover to the group G. That is, in this case, vertex $(M(\tilde{\Sigma}, \Sigma)) = G$

What about the moves and relation? None of the moves, $\mathbf{Z}, \mathbf{B_i}, \mathbf{F_{c,y}}, \mathbf{P_x T^c_{z,y}}$, make sense in this case since we do not have any cuts. But nevertheless, it is possible to define moves. Recall that in general, given a parameterization, f, of our G-cover, applying a move to f means we compose f with some standard automorphism of our "Standard Block", G-cover of S_n. In this case, the standard block is just$S_0 \times G$. So what are all the automorphism of the $S_0 \times G$? Again by the same argument as above we can identify the set of automorphism of $S_0 \times G$ with G. So moves of $M(\tilde{\Sigma}, \Sigma) = G$. More precisely we connect x to yx by an edge directed from x to yx. only relation here is precisely the relation satisfied by the group. Trivially this complex is connected and simply-connected (any closed loop starting from vertex x has the form $g_1 g_2 ... g_k x = x$ but then $g_1 g_2 ... g_k = 1$ which is a relation of our complex).

case 2: no cuts but at least one boundary circle

In this case, the base surface, Σ, is still simply-connected (remember that we always assume Σ has genus 0). So the G-cover of this is again $\Sigma \times G$. So all the argument above goes through and we see that vertex $(M(\tilde{\Sigma}, \Sigma)) = G$ and the edges of $M(\tilde{\Sigma}, \Sigma) = G$. And the complex $M(\tilde{\Sigma}, \Sigma)$ is connected and simply-connected.

case 3: at least one cut

This is the general situation. First recall the $\mathbf{P_x}$ relation:

(1) $\mathbf{P_x Z = Z P_x}$
(2) $\mathbf{P_x B_i = B_i P_x}$
(3) $\mathbf{P_x F_{c,y} = F_{c,yx^{-1}}(P_x \sqcup P_x)}$
(4) $\mathbf{P_x P_y = P_{xy}}$

and also the relation $\mathbf{T^c_{z,y} = F^{-1}_{c,z} F_{c,y}}$.

Now given a loop completely lie inside the fiber $\Pi^{-1}(a)$, we know from lemma 10 that this loop consist entirely of moves of the form $\mathbf{P_x}$ and $\mathbf{F^{-1}_{c_i,z} F_{c_i,y}} = \mathbf{T^{c_i}_{z,y}}$. First we translate all the move of the form $\mathbf{P_x}$ to the right and combine all the $\mathbf{P}$ move together to create a single $\mathbf{P}$ move. This can be done by the above three $\mathbf{P_x}$-relation. So we may assume that the loop looks like :

$$\mathbf{T^{c_1}_{z_{1,1},y_{1,1}} T^{c_1}_{z_{1,2},y_{1,2}} .... T^{c_1}_{z_{1,n_1},y_{1,n_1}} ...... T^{c_k}_{z_{k,1},y_{k,1}} T^{c_k}_{z_{k,2},y_{k,2}} .... T^{c_k}_{z_{k,n_k},y_{k,n_k}} P_x \sqcup P_x \sqcup .... \sqcup P_x}(b)$$

Here we choose b to be the starting vertex for our loop. Note that the indices for the $\mathbf{P}$ move are all same, which we denote by x. This must be the case for $\mathbf{T}$ move to make sense. Note also that that we gather all $\mathbf{T}$ move associated to a given cut c. This can be done since first of all, the associativity of cuts says that $\mathbf{F_{c_j,y} F_{c_i,z} = F_{c_i,z} F_{c_j,y}}$ whenever i is different from j. From this it follows

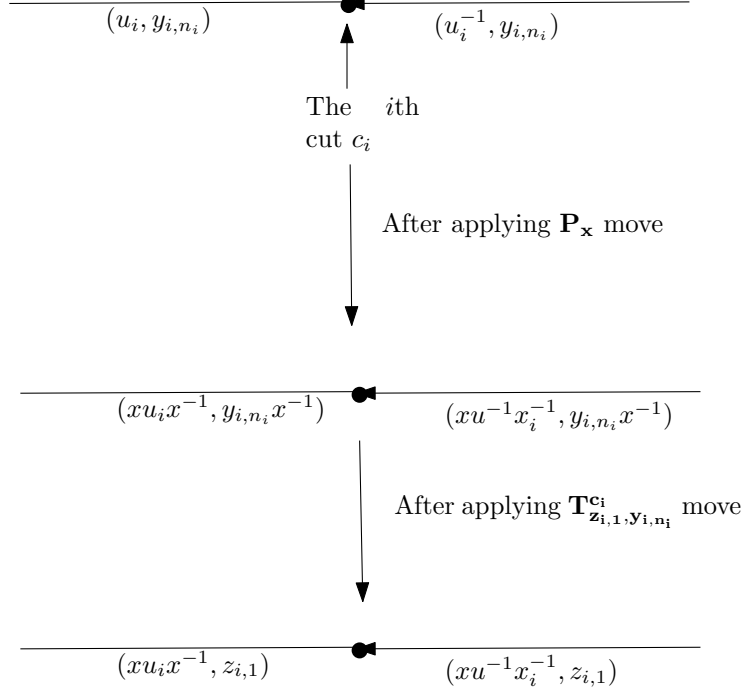(u_i, y_{i,n_i}) (u_i^{-1}, y_{i,n_i})

The ith
cut c_i

After applying $\mathbf{P_x}$ move

$(xu_i x^{-1}, y_{i,n_i} x^{-1})$ $(xu^{-1} x_i^{-1}, y_{i,n_i} x^{-1})$

After applying $\mathbf{T^{c_i}_{z_{i,1}, y_{i,n_i}}}$ move

$(xu_i x^{-1}, z_{i,1})$ $(xu^{-1} x_i^{-1}, z_{i,1})$

FIGURE 39.

that $\mathbf{T^{c_i}_{z_1, y_1} T^{c_j}_{z_2, y_2}} = \mathbf{T^{c_j}_{z_2, y_2} T^{c_i}_{z_1, y_1}}$ whenever i is different from j. Thus we can bring all $\mathbf{T}$ move associated to a given cut in one place. Now consider the ith cut, c_i. What can we say about $\mathbf{T^{c_i}_{z_{i,1}, y_{i,1}} T^{c_i}_{z_{i,2}, y_{i,2}} ..... T^{c_i}_{z_{i,n_i}, y_{i,n_i}}}$? We use the relation $\mathbf{T^c_{z,y}} = \mathbf{F^{-1}_{c,z} F_{c,y}}$ to conclude that $\mathbf{T^{c_i}_{z_{i,1}, y_{i,1}} T^{c_i}_{z_{i,2}, y_{i,2}} ..... T^{c_i}_{z_{i,n_i}, y_{i,n_i}}} = \mathbf{F^{-1}_{c_i, z_{i,1}} F_{c_i, y_{i,n_i}}}$. All the middle part will be $\mathbf{FF^{-1}} = \mathbf{1}$. So now our loop has the form

$$\mathbf{F^{-1}_{c_1, z_{1,1}} F_{c_1, y_{1,n_1}} F^{-1}_{c_2, z_{2,1}} F_{c_2, y_{2,n_2}} .... F^{-1}_{c_k, z_{k,1}} F_{c_k, y_{k,n_k}} P_x \sqcup P_x \sqcup .... \sqcup P_x}(b)$$

or

$$\mathbf{T^{c_1}_{z_{1,1}, y_{1,n_1}} T^{c_2}_{z_{2,1}, y_{2,n_2}} .... T^{c_k}_{z_{k,1}, y_{k,n_k}} P_x \sqcup P_x \sqcup .... \sqcup P_x}(b)$$

Now consider the ith cut, c_i. Say this cut is labeled by (u_i, y_{i,n_i}) for one component and by (u_i^{-1}, y_{i,n_i}) for the other component. When we apply $\mathbf{P_x \sqcup P_x \sqcup .... \sqcup P_x}$ move to b, this cut will be relabeled by $(xu_i x^{-1}, y_{i,n_i} x^{-1})$ and $(xu_i^{-1} x^{-1}, y_{i,n_i} x^{-1})$ respectively. See how we change the label for $\mathbf{P_x}$ move in section[7]. Then we apply $\mathbf{T^{c_i}_{z_{i,1}, y_{i,n_i}}}$ move to get the label $(xu_i x^{-1}, z_{i,1})$ and $(xu_i^{-1} x^{-1}, z_{i,1})$. See the sequence of picture on figure 39 for a visual description. But since it is a closed loop, we must have

$$xu_i x^{-1} = u_i \text{ and } z_{i,1} = y_{i,n_i}$$

as $i = 1...k$ Now we go back to the beginning and rewrite our original loop but this time we will move $\mathbf{P_x \sqcup P_x \sqcup .... \sqcup P_x}$ to all the way left. Recall the relation $\mathbf{P_x F_{c,y}} = \mathbf{F_{c, yx^{-1}} (P_x \sqcup P_x)}$. This will imply the following two relation:

$$\mathbf{P_x F_{c, yx}} = \mathbf{F_{c,y} (P_x \sqcup P_x)} \text{ and } \mathbf{P_x F^{-1}_{c, yx}} = \mathbf{F^{-1}_{c,y} (P_x \sqcup P_x)}$$
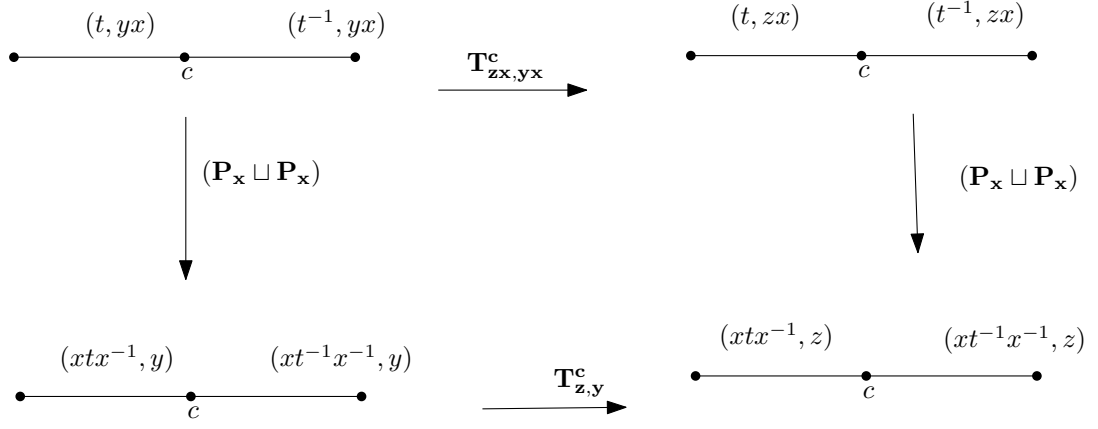
FIGURE 40. Visual description of what happens when we interchange $\mathbf{T}$ and $\mathbf{P}$ moves

These above two relation together with $\mathbf{T^c_{z,y}} = \mathbf{F^{-1}_{c,z}F_{c,y}}$ will imply the following:

$$\mathbf{T^c_{z,y}(P_x \sqcup P_x) = (P_x \sqcup P_x)T^c_{zx,yx}}$$

See the figure 40 for a visual description of the above relation. So when we move $\mathbf{P_x \sqcup P_x... \sqcup P_x})$ to the left, our loop will look like:

$$\mathbf{P_x \sqcup P_x \sqcup .... \sqcup P_x F^{-1}_{c_1,z_{1,1}x}F_{c_1,y_{1,n_1}x}F^{-1}_{c_2,z_{2,1}x}F_{c_2,y_{2,n_2}}x....F^{-1}_{c_k,z_{k,1}x}F_{c_k,y_{k,n_k}}x}(b)$$

or

$$\mathbf{P_x \sqcup P_x \sqcup .... \sqcup P_x T^{c_1}_{z_{1,1}x,y_{1,n_1}x}T^{c_2}_{z_{2,1}x,y_{2,n_2}}x....T^{c_k}_{z_{k,1}x,y_{k,n_k}}x}(b)$$

Similarly as before, we consider the ith cut, c_i, and label it with $(u_i, y_{i,n_i}x)$ for one component and $(u_i^{-1}, y_{i,n_i}x)$ for the other component. Now we apply $\mathbf{T^{c_i}_{z_{i,1}x,y_{i,n_i}x}}$ move first to get the label $(u_i, z_{i,1}x)$ and $(u_i^{-1}, z_{i,1}x)$ respectively. Now we need to apply $\mathbf{P_x \sqcup P_x \sqcup .... \sqcup P_x}$ move, and this will give us the label $(xu_ix^{-1}, z_{i,1})$ and $(xu_i^{-1}x^{-1}, z_{i,1})$. Since this is a closed loop, in particular this will imply $z_{i,1} = y_{i,n_i}x$ as $i = 1..k$. But we already know from above that $z_{i,1} = y_{i,n_i}$. So this means $x = 1$. That is $\mathbf{P_x = P_1}$ = identity or empty edge. Now our loops look like

$$\mathbf{F^{-1}_{c_1,z_{1,1}}F_{c_1,y_{1,n_1}}F^{-1}_{c_2,z_{2,1}}F_{c_2,y_{2,n_2}}....F^{-1}_{c_k,z_{k,1}}F_{c_k,y_{k,n_k}}}(b)$$

But since we already found out that $z_{i,1} = y_{i,n_i}$ as $i = 1..k$. So each $\mathbf{F^{-1}_{c_i,z_{i,1}}F_{c_i,y_{i,n_i}}} = 1$. So our loop is contractable. $\square$

7.2.4. *The complex $M(\Sigma)$.*

Lemma 13. *The complex $M(\Sigma)$ is connected and simply-connected. Here the edges of the complex are Z, B, F, define on the* [BK]. *Also the relation is defined on the same paper.*

Proof. This is exactly the [BK] is all about. So this paper is heavily depend on this paper. We will not repeat the proof here. Interested readers are referred to the paper [BK].

$\square$

7.2.5. *Proving part 3 of sec 9.1.* Let

$$b_1' \xrightarrow{f'} b_2'$$

and

$$b_1'' \xrightarrow{f''} b_2''$$

be two lifting of

$$a_1 \xrightarrow{f} a_2$$

Then we need to show that there is a path e_1, starting from b_1'' and end with b_1', and completely lie in $\Pi^{-1}(a_1)$ and a path e_2, starting from b_2'' and end with b_2', and completely lie in $\Pi^{-1}(a_2)$, so that the following relation hold:

$$e_2 f''(b_1'') = f' e_1(b_1'')$$

We know from lemma 8 that the vertex b_1'' and b_1' is connected by a sequence of move of the form $\mathbf{T}_{\mathbf{z},\mathbf{y}}^{\mathbf{c}}$ and $\mathbf{P}_{\mathbf{x}}$ and their inverses. We choose such a path from b_1'' to b_1'. Let us say the path is

$$(\mathbf{P}_{\mathbf{x_1}} \sqcup \mathbf{P}_{\mathbf{x_2}} \sqcup .... \sqcup \mathbf{P}_{\mathbf{x_k}})\mathbf{T}_{\mathbf{z_1},\mathbf{y_1}}^{\mathbf{c_1}} \mathbf{T}_{\mathbf{z_2},\mathbf{y_2}}^{\mathbf{c_2}}....\mathbf{T}_{\mathbf{z_k},\mathbf{y_k}}^{\mathbf{c_k}}(b_1'')$$

For later reference, we will denote $(\mathbf{P}_{\mathbf{x_1}} \sqcup \mathbf{P}_{\mathbf{x_2}} \sqcup .... \sqcup \mathbf{P}_{\mathbf{x_k}})\mathbf{T}_{\mathbf{z_1},\mathbf{y_1}}^{\mathbf{c_1}} \mathbf{T}_{\mathbf{z_2},\mathbf{y_2}}^{\mathbf{c_2}}....\mathbf{T}_{\mathbf{z_k},\mathbf{y_k}}^{\mathbf{c_k}} = L$. So this path from b_1'' to b_1' is just $L(b_1'')$. Note that this path $L(b_1'')$ completely lie inside the fiber $\Pi^{-1}(a_1)$. So we get our e_1 which is just $L(b_1'')$. Now depending on what kind of path the f is, we will have a different construction for the path e_2.

___case 1___: $f = Z$. In this case, we choose $e_2 = L(b_2'')$ more precisely, the path from b_2'' to b_2' is

$$e_2 = L(b_2'') = (\mathbf{P}_{\mathbf{x_1}} \sqcup \mathbf{P}_{\mathbf{x_2}} \sqcup .... \sqcup \mathbf{P}_{\mathbf{x_k}})\mathbf{T}_{\mathbf{z_1},\mathbf{y_1}}^{\mathbf{c_1}} \mathbf{T}_{\mathbf{z_2},\mathbf{y_2}}^{\mathbf{c_2}}....\mathbf{T}_{\mathbf{z_k},\mathbf{y_k}}^{\mathbf{c_k}}(b_2'')$$

Since in this case, both f' and f'' is the $\mathbf{Z}$ move, the commutativity that we want to show is

$$L\mathbf{Z}(b_1'') = \mathbf{Z}L(b_1'')$$

Recall $(\mathbf{P}_{\mathbf{x_1}} \sqcup \mathbf{P}_{\mathbf{x_2}} \sqcup .... \sqcup \mathbf{P}_{\mathbf{x_k}})\mathbf{T}_{\mathbf{z_1},\mathbf{y_1}}^{\mathbf{c_1}} \mathbf{T}_{\mathbf{z_2},\mathbf{y_2}}^{\mathbf{c_2}}....\mathbf{T}_{\mathbf{z_k},\mathbf{y_k}}^{\mathbf{c_k}} = L$. But the $\mathbf{Z}$ move commute with all the moves, so in particular, this means $L\mathbf{Z} = \mathbf{Z}L$ and we are done.

___case 2___: $f = B$. In this case, again we choose $e_2 = L(b_2'')$ more precisely, the path from b_2'' to b_2' is

$$e_2 = L(b_2'') = (\mathbf{P}_{\mathbf{x_1}} \sqcup \mathbf{P}_{\mathbf{x_2}} \sqcup .... \sqcup \mathbf{P}_{\mathbf{x_k}})\mathbf{T}_{\mathbf{z_1},\mathbf{y_1}}^{\mathbf{c_1}} \mathbf{T}_{\mathbf{z_2},\mathbf{y_2}}^{\mathbf{c_2}}....\mathbf{T}_{\mathbf{z_k},\mathbf{y_k}}^{\mathbf{c_k}}(b_2'')$$

Since in this case, both f' and f'' is the $\mathbf{B}$ move, the commutativity that we want to show is

$$L\mathbf{B}(b_1'') = \mathbf{B}L(b_1'')$$

here we suppress the indices for $\mathbf{B}$ since it is not important. Again recall

$$(\mathbf{P}_{\mathbf{x_1}} \sqcup \mathbf{P}_{\mathbf{x_2}} \sqcup .... \sqcup \mathbf{P}_{\mathbf{x_k}})\mathbf{T}_{\mathbf{z_1},\mathbf{y_1}}^{\mathbf{c_1}} \mathbf{T}_{\mathbf{z_2},\mathbf{y_2}}^{\mathbf{c_2}}....\mathbf{T}_{\mathbf{z_k},\mathbf{y_k}}^{\mathbf{c_k}} = L$$

But the $\mathbf{B}$ move also commute with the $\mathbf{T}$ and $\mathbf{P}$ moves (see the $\mathbf{B}$ relation). So in particular, this means $L\mathbf{B} = \mathbf{B}L$ and we are done.

___case 3___: $f = F$. This situation is little bit different from the above two. Here the F move for $M(\Sigma)$ and the $\mathbf{F}$ move for $M(\tilde{\Sigma}, \Sigma)$, both will remove a cut, say the

i-th cut. Then we choose

$$e_2 = (\mathbf{P_{x_1}} \sqcup \mathbf{P_{x_2}} \sqcup ... \sqcup \hat{\mathbf{P}}_{\mathbf{x_i}} \sqcup ... \sqcup \mathbf{P_{x_k}}) \underbrace{\mathbf{T_{z_1,y_1}^{c_1}} \mathbf{T_{z_2,y_2}^{c_2}} .... \mathbf{T_{z_k,y_k}^{c_k}}(b_2'')}_{\text{but we do not include } \mathbf{T_{z_i,y_i}^{c_i}}}$$

Here $\hat{\mathbf{P}}_{\mathbf{x_i}}$ means we do not include $\mathbf{P_{x_i}}$. then the commutativity that we want show is:

$$\mathbf{F}(\mathbf{P_{x_1}} \sqcup ...\mathbf{P_{x_k}})\mathbf{T_{z_1,y_1}^{c_1}} \mathbf{T_{z_2,y_2}^{c_2}} .... \mathbf{T_{z_k,y_k}^{c_k}} = (\mathbf{P_{x_1}} \sqcup ... \sqcup \hat{\mathbf{P}}_{\mathbf{x_i}} \sqcup ... \sqcup \mathbf{P_{x_k}}) \underbrace{\mathbf{T_{z_1,y_1}^{c_1}} \mathbf{T_{z_2,y_2}^{c_2}} .... \mathbf{T_{z_k,y_k}^{c_k}}\mathbf{F}}_{\text{but we do not include } \mathbf{T_{z_i,y_i}^{c_i}}}$$

This is very easy to show. We will give the argument anyway for completeness. We consider each cut one at a time. First consider the i th cut, the cut removed by $\mathbf{F}$. The left hand side of the above equation will apply a bunch of $\mathbf{T}$ and a $\mathbf{P_x}$ move to this cut first. But it does not matter, since at the end the $\mathbf{F}$ move will remove this cut. And the right hand side of the above euation will apply $\mathbf{F}$ move first and remove this i-th cut. Now consider the j-th cut where i and j are different. In this case, the $\mathbf{F}$ move does not have any effect (does not change the label) on the j-th cut. So both side of the above equation give rise to the same label on this j-th cut (we can just forget about the $\mathbf{F}$ move from both side since it does not have any effect). This finishes the proof.

7.2.6. *Proving part 4 of sec 9.1.* We need to show that Every loop in $M(\Sigma)$ can be lifted to a contractable loop in $M(\tilde{\Sigma}, \Sigma)$. First note that, it is enough to prove this for 2-cell or relations in $M(\Sigma)$. Because, then any loop in $M(\Sigma)$ can be break down to several 2-cell in $M(\Sigma)$, since $M(\Sigma)$ is simply-connected. We then lift each 2-cell to a contractable loop in $M(\tilde{\Sigma}, \Sigma)$. Then finally, we use part 3 of section 9.1, to arrive at our answer.
Now we will show that every 2-cell in $M(\Sigma)$ can be lifted to a contractable loop in $M(\tilde{\Sigma}, \Sigma)$. We first observe that all the 2-cell in $M(\Sigma)$, consist of Z, B and F move. And each 2-cell has a corresponding exact 2-cell in $M(\tilde{\Sigma}, \Sigma)$. For example, consider rotation axiom, the 2-cell in $M(\Sigma)$ is $Z^n = 1$ and the corresponding 2-cell in $M(\tilde{\Sigma}, \Sigma)$ is $\mathbf{Z}^n = 1$. Similarly for the braiding axiom, the 2-cell of $M(\Sigma)$ is $B_{\alpha,\gamma}B_{\alpha,\beta} = B_{\{\alpha\},\{\beta,\gamma\}}$ and $B_{\alpha,\gamma}B_{\beta,\gamma} = B_{\{\alpha\beta\},\{\gamma\}}$ and the corresponding 2-cell of $M(\tilde{\Sigma}, \Sigma)$ is $\mathbf{B}_{\alpha,\gamma}\mathbf{B}_{\alpha,\beta} = \mathbf{B}_{\{\alpha\},\{\beta,\gamma\}}$ and $\mathbf{B}_{\alpha,\gamma}\mathbf{B}_{\beta,\gamma} = \mathbf{B}_{\{\alpha\beta\},\{\gamma\}}$. The only exception to this rule is the dehn twist axiom. Dehn twist axiom for $M(\Sigma)$ is $ZB_{\alpha,\beta} = B_{\beta,\alpha}Z$ and the corresponding dehn twist axiom for $M(\tilde{\Sigma}, \Sigma)$ is $\mathbf{ZB}_{\alpha,\beta} = \mathbf{P_g}\mathbf{B}_{\beta,\alpha}\mathbf{Z}$, we have this extra $\mathbf{P_g}$ move appearing in the case of $M(\tilde{\Sigma}, \Sigma)$. But this will not be a problem. more precisely, we do the following:
If L is a 2-cell in $M(\Sigma)$, say starting at the vertex a, then we first pick any point on the fiber of a. Say $b \in \Pi^{-1}(a)$, it does not matter which point on the fiber we choose. Then we apply the corresponding 2-cell move of $M(\tilde{\Sigma}, \Sigma)$ to b and this will give a contractable loop in $M(\tilde{\Sigma}, \Sigma)$. For example if we have the 2-cell $Z^n(a)$ in $M(\Sigma)$ then a lifting of this 2-cell which is contractable is going to be $\mathbf{Z}^n(b)$ in $M(\tilde{\Sigma}, \Sigma)$. Look at the diagram on figure 41. For the dehn twist, it is not a big difference. If $ZB_{\alpha,\beta} = B_{\beta,\alpha}Z(a)$ is the 2-cell in $M(\Sigma)$ then $\mathbf{ZB}_{\alpha,\beta} = \mathbf{P_g}\mathbf{B}_{\beta,\alpha}\mathbf{Z}(b)$ will be the corresponding lifting in $M(\tilde{\Sigma}, \Sigma)$ which is of course contractible for the simple reason that $\mathbf{ZB}_{\alpha,\beta} = \mathbf{P_g}\mathbf{B}_{\beta,\alpha}\mathbf{Z}$ is a 2-cell in $M(\tilde{\Sigma}, \Sigma)$. See the diagram on

Here,
$b \in M(\tilde{\Sigma}, \Sigma)$
and
$b \in \Pi^{-1}(a)$
It does not
matter
what point we
choose
for $b \in \Pi^{-1}(a)$

α

γ (g_1, h_1) β

$\mathbf{Z}(b)$ (g_3, h_3) (g_2, h_2)

β

α (g_1, h_1) (g_2, h_2) γ

(g_3, h_3)

$\mathbf{Z}^2(b)$

Visualization
of
a

$\mathbf{Z}^3(b)$

γ

β (g_3, h_3) (g_1, h_1) α

(g_2, h_2)

the
lift-
ing
of
2-
cell
$Z^3 = 1$
in
$M(\tilde{\Sigma}, \Sigma)$

α

γ β

$Z(a)$

$Z^2(a)$

β γ

α

Here, $a \in M(\Sigma)$ Visualization
of
a

$Z^3(a)$

γ

β α

the
2-
cell
$Z^3 = 1$
in
$M(\Sigma)$
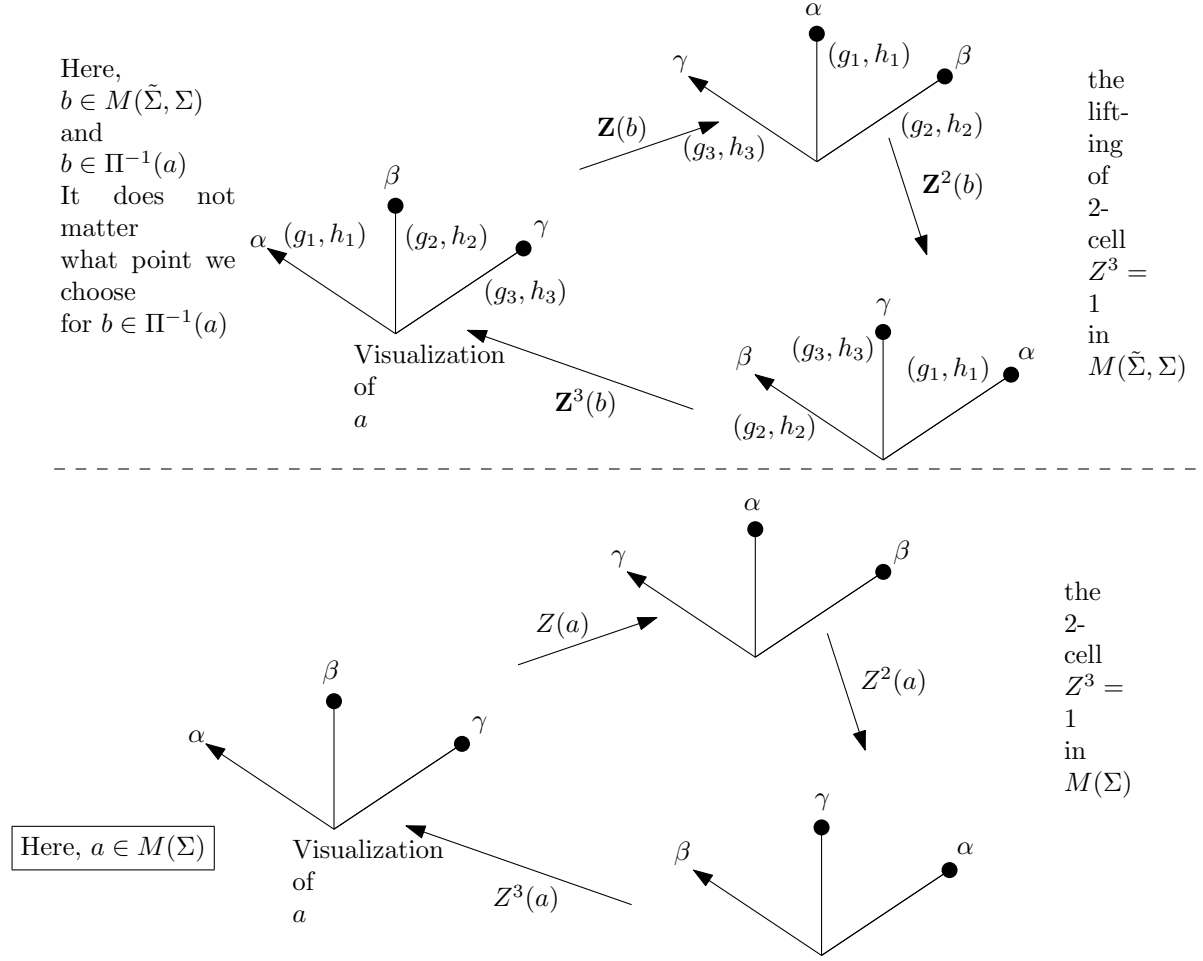
FIGURE 41. Here we take $n = 3$ for simplicity of drawing diagram

figure 42. So we satisfy all the condition of sec 9.1. This finishes the proof of our
main theorem.

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β

α

(g, h_1) $\mathbf{ZB}_{\alpha,\beta}(f)$ $(g^{-1}, h_2 g^{-1})$

β (g, h_1) α

$\mathbf{P_g B}_{\beta,\alpha}\mathbf{Z}(f)$

$(g^{-1}, h_2 g^{-1})$

Lifting of dehn twist
axiom of $M(\Sigma)$ to the
2-cell of dehn wist ax-
iom of $M(\tilde{\Sigma}, \Sigma)$

$\mathbf{B}_{\alpha,\beta}(f)$

α β

vertex b in
$M(\tilde{\Sigma}, \Sigma)$ (g, h_1) (g^{-1}, h_2)
where
$b \in \Pi^{-1}(a)$

α β

$(g, h_1 g)$ (g^{-1}, h_2)

$\mathbf{B}_{\beta,\alpha}\mathbf{Z}(f)$

β α

$\mathbf{Z}(f)$ (g^{-1}, h_2) (g, h_1)

α β

2-cell of dehn twist axiom $B_{\alpha,\beta}(a)$ $ZB_{\alpha,\beta}(a)$
in $M(\Sigma)$ α β

α β

vertex a
in $M(\Sigma)$ $Z(a)$ α β $B_{\beta,\alpha}Z(a)$

FIGURE 42.